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Geodesic vector fields and Eikonal equation on a Riemannian manifold

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Abstract

In this paper, we study the impact of geodesic vector fields (vector fields whose trajectories are geodesics) on the geometry of a Riemannian manifold. Since, Killing vector fields of constant lengths on a Riemannian manifold are geodesic vector fields, leads to the question of finding sufficient conditions for a geodesic vector field to be Killing. In this paper, we show that a lower bound on the Ricci curvature of the Riemannian manifold in the direction of geodesic vector field gives a sufficient condition for the geodesic vector field to be Killing. Also, we use a geodesic vector field on a 3-dimensional complete simply connected Riemannian manifold to find sufficient conditions to be isometric to a 3-sphere. We find a characterization of an Einstein manifold using a Killing vector field. Finally, it has been observed that a major source of geodesic vector fields is provided by solutions of Eikonal equations on a Riemannian manifold and we obtain a characterization of the Euclidean space using an Eikonal equation.

Keywords: Geodesic vector field; Eikonal equation; Killing vector field; Einstein manifold

1. Introduction

Recall that a geodesic vector field on a Riemannian manifold (M, g) is a vector field whose trajectories are geodesics. Thus, ξ is a geodesic vector field on the Riemannian manifold (M, g) if it satisfies the differential equation

$$\nabla_\xi \xi = 0, \tag{1}$$

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where ∇_X is the covariant derivative with respect to the Riemannian connection. Natural examples of geodesic vector fields are, the Reeb vector field on a Sasakian manifold (cf. [1]), Killing vector fields of constant length on a Riemannian manifold (cf. [2]), constant vector fields on the Euclidean space $\mathbb{R}^n$. Moreover, there are Killing vector fields, which are not geodesic vector fields, these examples are provided by Killing vector fields with non-constant lengths. For instance, the vector field $\xi = J\Psi$ on the Euclidean space $\mathbb{R}^{2n}$, where J is the standard complex structure and Ψ is the position vector field of the Euclidean space $\mathbb{R}^{2n}$, is a Killing vector field that is not a geodesic vector field. Similarly, there are geodesic vector fields, which are not Killing vector fields, for example the Reeb vector field on a 3-dimensional trans-Sasakian manifold (cf. [3]) and the Reeb vector field on a Kenmotsu manifold (cf. [4]) are geodesic vector fields which are not Killing vector fields.

Also, note that lot of geodesic vector fields come through physics and medical imaging, namely the solutions of Eikonal equation $\|\nabla f\| = 1$ on a Riemannian manifold (M, g) (cf. [5]). If a smooth function f is a solution of the Eikonal equation $\|\nabla f\| = 1$, then we get

$$\nabla_{\nabla f} \nabla f = 0,$$

that is, ∇f is a geodesic vector field. The distance function of a complete Riemannian manifold (M, g) satisfies Eikonal equation and therefore the gradient of the distance function is a geodesic vector field on (M, g). Also, it follows that the existence of Eikonal equation on a Riemannian manifold (M, g) implies that M is not compact.

Recall that a smooth vector field ξ on a Riemannian manifold (M, g) is said to be Killing if its local flow consists of local isometries of the Riemannian manifold (M, g). The presence of a nonzero Killing vector field on a compact Riemannian manifold constrains its geometry as well as topology, for instance, it does not allow the Riemannian manifold to have negative Ricci curvature and also if the Riemannian manifold has positive sectional curvature, then its fundamental group has a cyclic subgroup (cf. [2,6]). Apart from playing an important role in shaping geometry of a Riemannian manifold, Killing vector fields have applications in physics (cf. [7]) and therefore geodesic vector fields also have the scope of applications in physics. The geometry of Riemannian manifolds with Killing vector fields has been studied quite extensively (cf. [2,6,8–12]). It is worth mentioning that given a global gradient field ∇f is Killing if and only if it is parallel, and a characterization for these fields would be that the Hessian is zero, that is, $H_f = 0$.

Note that geodesic vector fields and Killing vector fields are closely related, though this relation gets weaker in the sense that if we consider two different vector fields (extensions of the same geodesic vector field ξ) satisfying $\nabla_\xi \xi = 0$, it may be possible that one is Killing and the other is not Killing as the first requires only what happens along the geodesic, while the other requires to take into account a neighborhood (cf. [13,14]).

Since, a Killing vector field on a Riemannian manifold of constant length is a geodesic vector field, a natural question is to seek conditions under which a geodesic vector field on a Riemannian manifold is a Killing vector field. In this paper, we find a characterization for a Killing vector field using a geodesic vector field (cf. Theorem 1). On a 3-dimensional sphere $\mathbb{S}^3(c)$ of constant curvature c, there is a Sasakian structure with Reeb vector field a geodesic vector field (cf. [1]), which leads to a question of finding a characterization of 3-sphere using a geodesic vector field. In this paper, we answer this question (cf. Theorem 2). Also, we find a characterization of an Einstein manifold using a Killing vector field on a connected Riemannian manifold, which improves a result in (cf. [8]). Finally, we use a geodesic vector field provided by an Eikonal equation on an n-dimensional complete Riemannian manifold (M, g) to find a characterization of the Euclidean space $\mathbb{R}^n$ (cf. Theorem 4).

2. Preliminaries

In this paper, we consider n-dimensional connected Riemannian manifold (M, g) unless differently specified. We denote by $\mathfrak{X}(M)$ the Lie algebra of smooth vector fields on M and by ∇_X the covariant derivative with respect to $X \in \mathfrak{X}(M)$, where ∇ is the Levi-Civita connection, which is a torsionfree metric connection (cf. [15]). A local orthonormal frame on M is a set of unit vectors $\{e_1, ..e_n\}$ defined on a neighborhood U of a point $p \in M$ satisfying $g(e_i, e_j) = \delta_{ij}$. For each point $p \in M$, we can choose a local orthonormal frame $\{e_1, ..e_n\}$ satisfying (cf. [15])

$$(\nabla e_i)(p) = 0.$$

Let ξ be a nonzero geodesic vector field on an n-dimensional Riemannian manifold (M, g). If η is the smooth 1-form dual to ξ, that is, $\eta(X) = g(X, \xi)$, then using the fact that the Lie derivative $\pounds_\xi g$ is a symmetric tensor, while the exterior derivative $d\eta$ being a smooth 2-form is a skew symmetric tensor, we get a symmetric operator A and a skew symmetric operator φ on (M, g) defined by

$$\frac{1}{2}\left(\pounds_\xi g\right)(X, Y) = g(AX, Y), \quad X, Y \in \mathfrak{X}(M) \tag{2}$$

and

$$\frac{1}{2} d\eta(X, Y) = g(\varphi X, Y), \quad X, Y \in \mathfrak{X}(M). \tag{3}$$

Now, using the following expressions for the Lie derivative $\pounds_\xi$ and the exterior derivative $d\eta$

$$\left(\pounds_\xi g\right)(X, Y) = g\left(\nabla_X \xi, Y\right) + g\left(\nabla_Y \xi, X\right), \ (d\eta)(X, Y) = g\left(\nabla_X \xi, Y\right) - g\left(\nabla_Y \xi, X\right),$$

we deduce that

$$\begin{aligned} \frac{1}{2} d\eta(X, Y) &= \frac{1}{2} g\left(\nabla_X \xi, Y\right) - \frac{1}{2} g\left(\nabla_Y \xi, X\right) \\ &= g\left(\nabla_X \xi, Y\right) - \frac{1}{2}\left(\pounds_\xi g\right)(X, Y). \end{aligned}$$

Using Eqs. (2), (3) in above equation, we get the following expression for the covariant derivative of the geodesic vector field ξ

$$\nabla_X \xi = AX + \varphi X, \quad X \in \mathfrak{X}(M). \tag{4}$$

Define a smooth function $f = \frac{1}{2}\|\xi\|^2$, which on using Eq. (4), gives

$$X(f) = g\left(\nabla_X \xi, \xi\right) = g\left(A\xi - \varphi\xi, X\right), \quad X \in \mathfrak{X}(M),$$

that is, the gradient ∇f of the function f is given by

$$\nabla f = A\xi - \varphi\xi. \tag{5}$$

Lemma 1. *Let (M, g) be an n- dimensional Riemannian manifold.*
(i) *If ξ is a geodesic vector field, then $A\xi = \frac{1}{2}\nabla f$, $\varphi\xi = -\frac{1}{2}\nabla f$.*
(ii) *If ξ is a Killing vector field, then $A = 0$, $\varphi\xi = -\nabla f$.*

Proof. (i) If ξ is a geodesic vector field, then Eq. (4) gives $A\xi + \varphi\xi = 0$, which in view of Eq. (5) gives the desired results.

(ii) If ξ is a Killing vector field, then Eqs. (2) and (5) give the desired results. □

Given a vector field $\boldsymbol{u} \in \mathfrak{X}(M)$ on an n -dimensional Riemannian manifold (M, g), its divergence div$\boldsymbol{u}$ is given by

$$\mathrm{div}\boldsymbol{u} = \sum g\left(\nabla_{e_i}\boldsymbol{u}, e_i\right),$$

where $\{e_1, ..e_n\}$ is a local orthonormal frame on M. Note that div$\boldsymbol{u}$ is the divergence with respect to the volume form of the Riemannian manifold (M, g). We denote by $(\nabla T)(X, Y)$ the covariant derivative of a $(1, 1)$ tensor field T and is given by

$$(\nabla T)(X, Y) = \nabla_X TY - T(\nabla_X Y), \quad X, Y \in \mathfrak{X}(M).$$

Now, we use Eq. (4) in computing the divergence of the vector field $A\xi$ as follows

$$\begin{aligned} \mathrm{div}(A\xi) &= \sum g\left(\nabla_{e_i} A\xi, e_i\right) = \sum \left(e_i g(\xi, Ae_i) - g\left(\xi, A\nabla_{e_i} e_i\right)\right) \\ &= \sum g(Ae_i + \varphi e_i, Ae_i) + \sum g\left(\xi, \nabla_{e_i} Ae_i - A\nabla_{e_i} e_i\right) \\ &= \|A\|^2 + g\left(\xi, \sum (\nabla A)(e_i, e_i)\right), \end{aligned} \tag{6}$$

where we have used

$$\|A\|^2 = \sum g(Ae_i, Ae_i), \text{ and } \sum g(Ae_i, \varphi e_i) = 0.$$

Similarly, we compute the divergence of the vector field $\varphi\xi$ and arrive at

$$\mathrm{div}(\varphi\xi) = -\|\varphi\|^2 - g\left(\xi, \sum (\nabla\varphi)(e_i, e_i)\right). \tag{7}$$

If ξ is a geodesic vector field on an n-dimensional Riemannian manifold (M, g), then Lemma 1(i) gives that $\mathrm{div}(A\xi) + \mathrm{div}(\varphi\xi) = 0$ and consequently, Eqs. (6), (7) yield

$$\|A\|^2 - \|\varphi\|^2 + g\left(\xi, \sum (\nabla A)(e_i, e_i) - (\nabla\varphi)(e_i, e_i)\right) = 0. \tag{8}$$

If ξ is a Killing vector field on an n-dimensional Riemannian manifold (M, g), then Eq. (2) gives $A = 0$ and Eq. (4) takes the form

$$\nabla_X \xi = \varphi X, \quad X \in \mathfrak{X}(M). \tag{9}$$

The curvature tensor field of a Riemannian manifold (M, g) is given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad X, Y, Z \in \mathfrak{X}(M).$$

For a Killing vector field ξ, using Eq. (9), we compute

$$R(X, Y)\xi = (\nabla\varphi)(X, Y) - (\nabla\varphi)(Y, X),$$

which together with the fact that the smooth 2-form $\Omega(X, Y) = g(\varphi X, Y)$ is closed, gives

$$g((\nabla\varphi)(X, Y), Z) + g(R(Y, Z)\xi, X) = 0.$$

Thus, we conclude that if ξ is a Killing vector field, then

$$(\nabla\varphi)(X, Y) = R(X, \xi)Y, \quad X, Y \in \mathfrak{X}(M). \tag{10}$$

On an n-dimensional Riemannian manifold (M, g), the Ricci tensor Ric is a symmetric tensor defined by

$$Ric(X, Y) = \sum g(R(e_i, X)Y, e_i), \quad X, Y \in \mathfrak{X}(M),$$

where $\{e_1,\ldots,e_n\}$ is a local orthonormal frame on M. The Ricci operator Q is a symmetric operator defined by

$$g(QX,Y)=Ric(X,Y),\quad X,Y\in\mathfrak{X}(M).$$

The scalar curvature S of a Riemannian manifold (M,g) is a smooth function defined by $S=tr\,Q$ (trace of the Ricci operator Q).

For a geodesic vector field ξ on a Riemannian manifold (M,g), using Eq. (4), we compute $R(X,Y)\xi$ and get

$$R(X,Y)\xi=(\nabla A)(X,Y)-(\nabla A)(Y,X)+(\nabla\varphi)(X,Y)-(\nabla\varphi)(Y,X).$$

We define a smooth function $h=tr\,A$ (the trace of the symmetric operator A), and use above equation to conclude

$$\begin{aligned}X(h)&=\sum g\left((\nabla A)(X,e_i),e_i\right)\\&=\sum g\left(R(X,e_i)\xi+(\nabla A)(e_i,X)-(\nabla\varphi)(X,e_i)+(\nabla\varphi)(e_i,X).e_i\right)\\&=-Ric(X,\xi)+g\left(X,\sum\left((\nabla A)(e_i,e_i)-(\nabla\varphi)(e_i,e_i)\right)\right),\end{aligned}$$

where we used the symmetry of the operator A and the skew symmetry of the operator φ. Thus, the gradient ∇h of the function h is given by

$$\nabla h=-Q(\xi)+\sum(\nabla A)(e_i,e_i)-\sum(\nabla\varphi)(e_i,e_i).\tag{11}$$

Given a smooth function F on an n-dimensional Riemannian manifold (M,g), its Laplacian ΔF is defined by $\Delta F=\operatorname{div}(\nabla F)$. Also, for a smooth vector field $X\in\mathfrak{X}(M)$, the Laplacian ΔX is a smooth vector field defined by

$$\Delta X=\sum\left(\nabla_{e_i}\nabla_{e_i}X-\nabla_{\nabla_{e_i}e_i}X\right),$$

where $\{e_1,\ldots,e_n\}$ is a local orthonormal frame on M.

3. Impact of geodesic vector fields

We prove the first result of this paper, which characterizes Killing vector fields.

Theorem 1. *Let ξ be a geodesic vector field on an n-dimensional connected Riemannian manifold (M,g). Then ξ is a Killing vector field of constant length if and only if Ricci curvature satisfies*

$$Ric(\xi,\xi)\geq\frac{1}{4}\|d\eta\|^2-\frac{1}{4n}\left(tr\left(\pounds_\xi g\right)\right)^2-\frac{1}{2}\xi\left(tr\left(\pounds_\xi g\right)\right),$$

where η is smooth 1-form dual to ξ, $\pounds_\xi g$ is the Lie derivative of metric g with respect to ξ and $tr\left(\pounds_\xi g\right)$ is the trace of the symmetric operator $\pounds_\xi g$.

Proof. Suppose ξ is a geodesic vector field and the given condition holds, that is,

$$Ric(\xi,\xi)\geq\|\varphi\|^2-\frac{1}{n}\|A\|^2-\xi(h).\tag{12}$$

Now, using Eqs. (8), (11), we get

$$\|A\|^2-\|\varphi\|^2+Ric(\xi,\xi)+\xi(h)=0,$$

that is,

$$\left(\|A\|^2-\frac{1}{n}h^2\right)+\left(Ric(\xi,\xi)-\|\varphi\|^2+\frac{1}{n}h^2+\xi(h)\right)=0.$$

Thus, using the inequality (12) and the Schwarz inequality $\|A\|^2\geq\frac{1}{n}(trA)^2$ in above equation, we conclude that

$$\|A\|^2=\frac{1}{n}h^2,\quad Ric(\xi,\xi)=\|\varphi\|^2-\frac{1}{n}\|A\|^2-\xi(h). \tag{13}$$

However, the equality in Schwarz inequality holds if and only if $A=\frac{h}{n}I$, which gives $A\xi=\frac{h}{n}\xi$, and combining it with $A\xi+\varphi\xi=0$, leads to $h\xi+\varphi\xi=0$. The last equation on taking the inner product with ξ, gives

$$h\|\xi\|^2=0$$

and as ξ is a nonzero vector field on the connected manifold M, we get $h=0$ and consequently, Eq. (13) now reads

$$A=0,\quad Ric(\xi,\xi)=\|\varphi\|^2.$$

Thus, Lemma 1(i), gives f is a constant and $\varphi\xi=0$, consequently, Eq. (4) gives $\left(\pounds_\xi g\right)(X,Y)=0$, that is, ξ is a Killing vector field. Moreover, $\varphi\xi=0$ and Eq. (4) insures that ξ has constant length.

Conversely, if ξ is a Killing vector field of constant length, then using Eq. (9) in $X\left(\|\xi\|^2\right)=0$, we get $\varphi\xi=0$, and consequently, Eq. (9), gives $\nabla_\xi\xi=0$, that is, ξ is a geodesic vector field. Finally, using Eq. (10), we conclude

$$\sum(\nabla\varphi)(e_i,e_i)=-Q\xi,$$

that is,

$$Ric(\xi,\xi)=-g\left(\xi,\sum(\nabla\varphi)(e_i,e_i)\right).$$

Now, using Eq. (7) with $\varphi\xi=0$ in above equation, we get

$$Ric(\xi,\xi)=\|\varphi\|^2,$$

which provides the inequality in the statement, as $\left(\pounds_\xi g\right)=0$. □

Note that a 3-dimensional sphere $\mathbb{S}^3(c)$ admits a Sasakian structure with Reeb vector field ξ a geodesic vector field (cf. [1]), that leads to a question of finding conditions on a 3-dimensional complete simply connected Riemannian manifold (M,g) that admits a geodesic vector field to be isometric to $\mathbb{S}^3(c)$. We prove the following theorem, which gives a characterization for the sphere $\mathbb{S}^3(c)$ using a geodesic vector field.

Theorem 2. *Let ξ be a geodesic vector field of constant length on a 3-dimensional complete simply connected conformally flat Riemannian manifold (M,g) with sectional curvatures of plane sections containing ξ positive. Then ξ satisfies*

$$\text{(i) } \Delta\xi=-\lambda\xi,\quad \text{(ii) } \frac{1}{4}\|d\eta\|^2\geq\lambda\|\xi\|^2-\frac{1}{12}\left(tr\left(\pounds_\xi g\right)\right)^2,$$

where λ is a positive constant, η is smooth 1-form dual to ξ, $\pounds_\xi g$ is the Lie derivative of metric g with respect to ξ; if and only if (M,g) is isometric to the 3-sphere $\mathbb{S}^3\left(\frac{\lambda}{2}\right)$.

Proof. Let ξ be a geodesic vector field of constant length on a 3-dimensional complete simply connected Riemannian manifold (M, g) with sectional curvatures of the plane sections containing ξ positive and satisfying conditions (i) and (ii). Note that the definition of a geodesic vector field and Eq. (4) gives $A\xi + \varphi\xi = 0$ and since $\|\xi\|$ is a constant gives $A\xi - \varphi\xi = 0$, consequently, we have $A\xi = \varphi\xi = 0$. Using Eq. (4) in computing $\Delta\xi$, we get

$$\Delta\xi = \sum (\nabla A)(e_i, e_i) + \sum (\nabla\varphi)(e_i, e_i) = -\lambda\xi, \tag{14}$$

which in view of Eqs. (6) and (7), yields

$$\|A\|^2 + \|\varphi\|^2 = \lambda \|\xi\|^2 .$$

Thus, we have

$$\left(\|A\|^2 - \frac{1}{3}h^2\right) + \left(\|\varphi\|^2 + \frac{1}{3}h^2 - \lambda \|\xi\|^2\right) = 0,$$

where $h = trA$. Using the condition (ii) in above equation and the Schwarz inequality, we get

$$A = \frac{h}{3}I \text{ and } \|\varphi\|^2 = \lambda \|\xi\|^2 - \frac{1}{3}h^2.$$

Now, using $A\xi = 0$ in above equation and the fact that ξ is nonzero, gives $h = 0$ and consequently,

$$A = 0, \quad \|\varphi\|^2 = \lambda \|\xi\|^2 .$$

In view of above equation, Eq. (4) takes the form

$$\nabla_X\xi = \varphi X, \quad X \in \mathfrak{X}(M),$$

which is Eq. (9) ensuring that ξ a Killing vector field. Using $Q(\xi) = \sum R(\xi, e_i)e_i$ and Eq. (14) with $A = 0$, in Eq. (10), we get

$$Q(\xi) = -\lambda\xi. \tag{15}$$

Since, ξ is a Killing vector field, its flow consists of isometries of the Riemannian manifold (M, g) and the scalar curvature S and the Ricci operator Q of M are both invariant under the flow, that is, we have

$$\left(\pounds_\xi S\right) = 0, \quad \left(\pounds_\xi Q\right) = 0,$$

which in view of Eq. (9), gives

$$\xi(S) = 0, \quad (\nabla Q)(\xi, X) = \varphi QX - Q\varphi X, \quad X \in \mathfrak{X}(M). \tag{16}$$

Since, (M, g) is a 3-dimensional conformally flat Riemannian manifold, we have (cf. [15,16])

$$(\nabla Q)(X, Y) - (\nabla Q)(Y, X) = \frac{1}{4}\{X(S)Y - Y(S)X\}, \quad X, Y \in \mathfrak{X}(M),$$

which on taking $Y = \xi$ and using Eq. (16), leads to

$$(\nabla Q)(X, \xi) - \varphi QX + Q\varphi X = \frac{1}{4}X(S)\xi.$$

Using Eqs. (9) and (15), we get $\nabla_X Q\xi = \lambda\varphi X$ and consequently, above equation implies

$$\varphi(QX - \lambda X) = -\frac{1}{4}X(S)\xi.$$

Taking the inner product in above equation with ξ and using $\varphi\xi = 0$, gives $X(S) = 0$ for all X, proving that the scalar curvature S is a constant and thus, we have

$$\varphi\left(QX - \lambda X\right) = 0, \quad X \in \mathfrak{X}(M) \,. \tag{17}$$

Also, Eqs. (9), (10) and $\varphi\xi = 0$, imply $R(X,\xi)\xi = -\varphi^2 X$, that is,

$$R(X,\xi;\xi,X) = \|\varphi X\|^2, \quad X \in \mathfrak{X}(M). \tag{18}$$

As the sectional curvatures of plane sections containing ξ are positive, from above equation we conclude that $\|\varphi X\| > 0$ for all nonzero X orthogonal to ξ. Note that Eq. (15) implies that $g(QX - \lambda X, \xi) = 0$, that is, the vector field $QX - \lambda X$ is orthogonal to ξ and Eqs. (17) and (18) lead to

$$R(QX - \lambda X, \xi; \xi, QX - \lambda X) = 0, \quad X \in \mathfrak{X}(M).$$

Thus, as the sectional curvatures of plane sections containing ξ are positive, above equation leads to

$$QX = \lambda X, \quad X \in \mathfrak{X}(M).$$

Now using the above outcome as $Ric(X,Y) = \lambda g(X,Y)$, in the expression for the curvature tensor of conformally flat 3-dimensional Riemannian manifold (M,g), we get (cf. [15], (cf. [16])),

$$R(X,Y;Z,W) = \frac{1}{2}\left(4\lambda - S\right)\left\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\right\}.$$

However, as $Ric(X,Y) = \lambda g(X,Y)$, we have $S = 3\lambda$ and consequently, above equation implies that (M,g) is a complete simply connected Riemannian manifold of constant positive sectional curvature $\frac{\lambda}{2}$ and hence, is isometric to the sphere $\mathbb{S}^3\left(\frac{\lambda}{2}\right)$.

The converse is trivial as the sphere $\mathbb{S}^3\left(\frac{\lambda}{2}\right)$ has Sasakian structure (cf. [1]) with Reeb vector field ξ being Killing is a geodesic vector field that satisfies all the requirements of the hypothesis. □

Recall that from Eq. (15) onwards in the proof of Theorem 2, all the steps work for a Killing vector field on a 3-dimensional Riemannian manifold, thus removing the restrictions in the hypothesis of Theorem 2 which are responsible for transforming a geodesic vector field into a Killing vector field, we get the following (A similar result is proved in [8], with ξ is an eigenvector of Q instead of Δ):

Corollary. *Let ξ be a Killing vector field of constant length on a 3-dimensional complete simply connected conformally flat Riemannian manifold (M,g) with sectional curvatures of plane sections containing ξ positive. Then ξ satisfies $\Delta\xi = -\lambda\xi$, where λ is a positive constant, if and only if (M,g) is isometric to the 3-sphere $\mathbb{S}^3\left(\frac{\lambda}{2}\right)$.*

Recall that a (1, 1) tensor field T on a Riemannian manifold (M,g) is said to be a Codazzi tensor if it satisfies

$$\left(\nabla T\right)(X,Y) = \left(\nabla T\right)(Y,X), \quad X,Y \in \mathfrak{X}(M).$$

In the following result, we use a nonzero Killing vector field ξ of constant length on a connected Riemannian manifold (M,g) and a weaker condition than Codazzi tensor on the Ricci operator Q to find the following characterization of an Einstein manifold (this result is an improvement of Theorem 4.1 in [8]).

Theorem 3. *Let ξ be a nonzero Killing vector field of constant length on an n-dimensional connected Riemannian manifold (M, g) with sectional curvatures of the plane sections containing ξ positive and Ricci curvature in the direction of ξ a constant. Then (M, g) is an Einstein manifold if and only if the Ricci operator Q satisfies*

$$(\nabla Q)(X, \xi) = (\nabla Q)(\xi, X), \quad X \in \mathfrak{X}(M).$$

Proof. Suppose ξ is a nonzero Killing vector field of constant length on the Riemannian manifold (M, g) satisfying the conditions of the hypothesis and the Ricci operator Q satisfies

$$(\nabla Q)(X, \xi) = (\nabla Q)(\xi, X). \tag{19}$$

As the Ricci curvature in the direction of ξ is a constant, we have $Ric(\xi, \xi) = \lambda \|\xi\|^2$, where λ is a constant. Then, we can express the vector field $Q(\xi)$ as follows

$$Q(\xi) = \lambda\xi + \mathbf{u}, \tag{20}$$

where $\mathbf{u}$ is a vector field orthogonal to ξ.

Note that the local flow of a Killing vector field consists of local isometries. Thus, we have $\left(\pounds_\xi Q\right) = 0$, which on using Eq. (9), gives

$$(\nabla Q)(\xi, X) = \varphi Q X - Q\varphi X, \quad X \in \mathfrak{X}(M). \tag{21}$$

Now, using Eqs. (9), (19) and (21), we get

$$\nabla_X Q\xi = \varphi Q X.$$

Thus, Eq. (20) in view of above equation and Eq. (9), gives

$$\varphi(QX - \lambda X) = \nabla_X \mathbf{u}. \tag{22}$$

Note that ξ being of constant length, $\varphi\xi = 0$ and consequently, above equation implies $g(\nabla_X \mathbf{u}, \xi) = 0$, that is, $g(\mathbf{u}, \varphi X) = 0$. Thus, $\varphi(\mathbf{u}) = 0$ and Eq. (10), gives

$$R(\mathbf{u}, \xi)\xi = -\varphi^2\mathbf{u} = 0,$$

that is, the vector field $\mathbf{u}$ orthogonal to ξ satisfies $R(\mathbf{u}, \xi; \xi, \mathbf{u}) = 0$. However, the sectional curvatures of the plane sections containing ξ are positive, which implies $\mathbf{u} = 0$ and consequently, Eq. (22), gives $\varphi(QX - \lambda X) = 0$, $X \in \mathfrak{X}(M)$. Note that as $\mathbf{u} = 0$, Eq. (20) reads as: $Q(\xi) = \lambda\xi$ and we have $g(QX - \lambda X, \xi) = g(Q(\xi) - \lambda\xi, X) = 0$, that is, the vector field $QX - \lambda X$ is orthogonal to ξ and the sectional curvature in view of Eq. (10) is given by

$$R(QX - \lambda X, \xi; \xi, QX - \lambda X) = \|\varphi(QX - \lambda X)\|^2 = 0.$$

Thus, $QX - \lambda X = 0$ for all X and this proves that (M, g) is an Einstein manifold. The converse is trivial. □

4. Eikonal equation

Recall that an Eikonal equation $\|\nabla f\| = 1$ on a Riemannian manifold (M, g) gives the vector field ∇f, which is a geodesic vector field. Note that the distance function r on the Euclidean space $\mathbb{R}^n$ is a smooth function on $\mathbb{R}^n - \{0\}$ and satisfies the Eikonal equation $\|\nabla r\| = 1$ and also satisfies $r\Delta r = n - 1$. This raises the question, whether the solution f of the Eikonal equation $\|\nabla f\| = 1$ on an n-dimensional complete connected Riemannian manifold (M, g) satisfying $f\Delta f = n - 1$, makes the Riemannian manifold isometric to the

Euclidean space $\mathbb{R}^n$? We use the tools developed for geodesic vector fields on a Riemannian manifold to answer this question and indeed obtain the following simple characterization of the Euclidean space $\mathbb{R}^n$.

Theorem 4. *Let f be a solution of the Eikonal equation $\|\nabla f\| = 1$ on an n- dimensional complete connected Riemannian manifold (M, g) with Ricci curvature in the direction of ∇f non-negative. Then f satisfies $f\Delta f = n-1$, if and only if (M, g) is isometric to the Euclidean space $\mathbb{R}^n$.*

Proof. Suppose (M, g) is an n-dimensional complete and connected Riemannian manifold that admits the solution f of the Eikonal equation $\|\nabla f\| = 1$ and the solution satisfies

$$f\Delta f = n - 1. \tag{23}$$

Then it follows that $\xi = \nabla f$ is a unit geodesic vector field that is closed and consequently, $\varphi = 0$ in Eq. (4) and as ξ is a geodesic vector field equation (4) gives $A\xi = 0$. Moreover by Eq. (4), we see that $A = A_f$, the Hessian operator of the function f and therefore $h = trA = \Delta f$ holds. Now, Eq. (11), gives

$$Q(\xi) = \sum (\nabla A)(e_i, e_i) - \nabla(\Delta f),$$

that is,

$$Ric(\xi, \xi) = g\left(\xi, \sum (\nabla A)(e_i, e_i)\right) - \xi(\Delta f). \tag{24}$$

Also, using Eq. (6), and $A\xi = 0$, we get

$$\|A\|^2 = -g\left(\xi, \sum (\nabla A)(e_i, e_i)\right). \tag{25}$$

Now, define a function $\psi = \frac{1}{2}f^2$, which is a nonconstant function and has gradient $\nabla\psi = f\nabla f = f\xi$, and using Eq. (4), we find the Hessian operator A_ψ of the function ψ, given by

$$A_\psi(X) = X(f)\xi + fAX, \quad X \in \mathfrak{X}(M). \tag{26}$$

The above equation gives $trA_\psi = \xi(f) + f\Delta f = \|\nabla f\|^2 + n - 1 = n$, and

$$\|A_\psi\|^2 = 1 + f^2\|A\|^2. \tag{27}$$

Thus, we have

$$\|A_\psi\|^2 - \frac{1}{n}\left(trA_\psi\right)^2 = 1 + f^2\|A\|^2 - n,$$

which in view of Eqs. (24) and (25), leads to

$$\|A_\psi\|^2 - \frac{1}{n}\left(trA_\psi\right)^2 = 1 - n - f^2\{Ric(\xi, \xi) + \xi(\Delta f)\}.$$

Observe that $f\xi(\Delta f) = \xi(f\Delta f) - \Delta f\xi(f) = -\Delta f\|\nabla f\|^2 = -\Delta f$ and consequently, above equation takes the form

$$\|A_\psi\|^2 - \frac{1}{n}\left(trA_\psi\right)^2 = 1 - n - f^2Ric(\xi, \xi) + f\Delta f = -f^2Ric(\xi, \xi).$$

Using the hypothesis that $Ric(\xi, \xi) \geq 0$ and the Schwarz inequality in above equation, we get

$$\|A_\psi\|^2 = \frac{1}{n}\left(trA_\psi\right)^2 \quad \text{and} \quad Ric(\xi, \xi) = 0.$$

However, in Schwarz inequality $\left\|A_\psi\right\|^2 \geq \frac{1}{n}\left(trA_\psi\right)^2$, the equality holds if and only if $A_\psi = \mu I$ for some smooth function μ. Since, $trA_\psi = n$, we get $\mu = 1$. Hence, the Hessian H_ψ of the nonconstant function ψ on the complete and connected Riemannian manifold (M, g), satisfies

$$H_\psi(X, Y) = g(X, Y), \quad X, Y \in \mathfrak{X}(M),$$

which proves that (M, g) is isometric to the Euclidean space $\mathbb{R}^n$ (cf. [17]).

The converse is trivial as the distance function f on the Euclidean space $\mathbb{R}^n$ satisfies all the requirements in the statement. □

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