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Conformal vector fields on submanifolds of a Euclidean space

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Abstract. In this paper, we investigate n-dimensional immersed compact submanifold M of a Euclidean space R^{n+p}, with the immersion $\psi : M \to R^{n+p}$, where the tangential component ψ^T of ψ is a conformal vector field. A characterization of n-sphere in the Euclidean space R^{n+p} is obtained. Also conditions under which ψ^T is a conformal vector field in the general case and those in the special case where the submanifold has flat normal connection and $p = 2$ are obtained as well.

1. Introduction

Given an immersed n-dimensional submanifold M of a Euclidean space $(R^{n+p}, \langle , \rangle)$, where $\langle , \rangle$ is the Euclidean metric, one of the important questions is to find conditions under which the submanifold M lies on the hypersphere $S^{n+p-1}(c)$ of the Euclidean space R^{n+p}. This question has been studied in [ALO07], [ALO02], [ALOD02]. Recall that a smooth vector field ξ on a Riemannian manifold (M, g) is said to be a conformal vector field if its flow consists of conformal transformations of the Riemannian manifold (M, g) and it is equivalent to the requirement that the vector field ξ satisfies

$$\pounds_\xi g = 2\rho g,$$

where $\pounds_\xi$ is the Lie derivative with respect to the vector field ξ, and ρ is a smooth function on M, called the potential function of the conformal vector

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field ξ. Conformal vector fields have been used to characterize spheres among compact Riemannian manifolds (cf. [DES12], [DES08], [DES10]). If M is an n-dimensional immersed submanifold of the Euclidean space R^{n+p} with the immersion $\psi : M \to R^{n+p}$, then treating ψ as the position vector field of points of M, we can express it as

$$\psi = \psi^T + \psi^\perp,$$

where ψ^T is the tangential component of ψ to M, and $\psi^\perp$ is the normal component of ψ. Thus, we get a globally defined vector field ψ^T on the submanifold M, which might be either a Killing vector field or a conformal vector field. However, the covariant derivative of ψ^T being symmetric (see Section 2), asking ψ^T be a Killing vector field, will not yield interesting geometry. Therefore, it is a natural question to find conditions under which the vector field ψ^T is a conformal vector field on M, as well as to study the geometry of the submanifold for which the vector field ψ^T is a conformal vector field. In this paper, we address these questions. It is interesting to note that in the case when ψ^T is a nonzero conformal vector field on the compact submanifold M, under suitable restrictions on the Ricci curvatures, the submanifold is shown to be isometric to the sphere $S^n(c)$ of constant curvature c (cf. Theorem 3.1). We also find conditions under which the vector field ψ^T is a conformal vector field on the submanifold M (cf. Theorems 3.2 and 4.1). Finally, we use the conformal vector field associated to the normal component $\psi^\perp$ on the submanifold M to find a necessary and sufficient condition for the submanifold to lie on the hypersphere $S^{n+p-1}(c)$ (cf. Theorem 3.3).

2. Preliminaries

Let M be an n-dimensional submanifold of the Euclidean space R^{n+p} with immersion $\psi : M \to R^{n+p}$. We denote by $\langle , \rangle$ and $\overline{\nabla}$ the Euclidean metric and the Euclidean connection, respectively, on R^{n+p}, we also denote by g and ∇ the induced metric and the Riemannian connection on the submanifold M. Then, we have the following equations for the submanifold M (cf. [CHE83]):

$$\overline{\nabla}_X Y = \nabla_X Y + h\left(X, Y\right), \quad \overline{\nabla}_X N = -A_N X + \nabla^\perp_X N \tag{2.1}$$

$X, Y \in \mathfrak{X}\left(M\right)$, $N \in \Gamma\left(\Lambda\right)$, where $\mathfrak{X}\left(M\right)$ is the Lie algebra of smooth vector fields on M, $\Gamma\left(\Lambda\right)$ is the space of smooth sections of the normal bundle Λ of M, h is the second fundamental form, A_N is the Weingarten map with respect to the normal $N \in \Gamma(\Lambda)$ which is related to the second fundamental form h by

$$g\left(A_N X, Y\right) = g\left(h\left(X, Y\right), N\right), \quad X, Y \in \mathfrak{X}\left(M\right),$$

and $\nabla^{\perp}$ is the connection in the normal bundle Λ. We also have the Gauss equation

$$R(X,Y)Z = A_{h(Y,Z)}X - A_{h(X,Z)}Y, \quad X,Y,Z \in \mathfrak{X}(M), \tag{2.2}$$

where $R(X,Y)Z,\ X,Y,Z \in \mathfrak{X}(M)$ is the curvature tensor field of the submanifold M. The Ricci tensor field of M is given by

$$\operatorname{Ric}(X,Y) = ng(h(X,Y),H) - \sum_{i=1}^{n} g(h(X,e_i),h(Y,e_i)), \tag{2.3}$$

where $\{e_1,...,e_n\}$ is a local orthonormal frame on M, and

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i,e_i)$$

is the mean curvature vector field.

The Ricci operator Q is the symmetric operator defined by

$$\operatorname{Ric}(X,Y) = g(Q(X),Y), \quad X,Y \in \mathfrak{X}(M).$$

If we express $\psi = \psi^T + \psi^{\perp}$, where $\psi^T \in \mathfrak{X}(M)$ is the tangential component and $\psi^{\perp} \in \Gamma(\Lambda)$ is the normal component of ψ, and if we denote by $B = A_{\psi^{\perp}}$ the Weingarten map with respect to the normal vector field $\psi^{\perp}$, then using equation (2.1), we get

$$\nabla_x \psi^T = X + BX, \quad \nabla^{\perp}_X \psi^{\perp} = -h\left(X,\psi^T\right), \quad X,Y \in \mathfrak{X}(M). \tag{2.4}$$

We use the mean curvature vector field H to define a smooth function $F : M \to R$ on the submanifold M by $F = \left\langle H,\psi^{\perp}\right\rangle$. Now, for an n-dimensional submanifold $\psi : M \to R^{n+p}$, and a local orthonormal frame $\{e_1,...,e_n\}$ on M, we have

$$\begin{aligned}\operatorname{div}\psi^T &= \sum_{i=1}^{n}\left\langle \nabla_{e_i}\psi^T, e_i\right\rangle = \sum_{i=1}^{n}\left\langle e_i + A_{\psi^{\perp}}e_i, e_i\right\rangle \\ &= n + \sum_{i=1}^{n}\left\langle h(e_i,e_i),\psi^{\perp}\right\rangle = n + n\left\langle H,\psi^{\perp}\right\rangle = n(1+F),\end{aligned}$$

that is,

$$\operatorname{div}\psi^T = n(1+F). \tag{2.5}$$

We have the following Lemmas:

Lemma 2.1 (Hsiung–Minkowski formula). *Let M be an n-dimensional compact submanifold of the Euclidean space R^{n+p}. Then*

$$\int_M (1+F)dv = 0.$$

Lemma 2.2 ([ALO07]). *Let M be an n-dimensional submanifold of R^{n+p}. Then the tensor field B satisfies*

(i) $\operatorname{Tr} B = nF$;

(ii) $(\nabla B)(X,Y) - (\nabla B)(Y,X) = R(X,Y)\psi^T$;

(iii) $\sum_{i=1}^n (\nabla B)(e_i,e_i) = n\nabla F + Q\left(\psi^T\right)$;

where $(\nabla B)(X,Y) = \nabla_X BY - B\nabla_X Y \ X,Y \in \mathfrak{X}(M)$.

Lemma 2.3 ([ALO07]). *Let $\psi : M \to R^{n+p}$ be an n-dimensional compact submanifold. Then a necessary and sufficient condition for $\psi(M) \subset S^{n+p-1}(c)$ is that $\psi^T = 0$ and $F = -1$.*

Definition 2.1. A smooth vector field ξ on a Riemannian manifold (M,g) is said to be a conformal vector field if there exists a smooth function ρ on M that satisfies $\pounds_\xi g = 2\rho g$, ρ called a potential function, where $\pounds_\xi g$ is the Lie derivative of g with respect to ξ. We say that ξ is a non-trivial conformal vector field if the potential function ρ is not a constant. A conformal vector field ξ is said to be a gradient conformal vector field if $\xi = \nabla f$, for a smooth function f on M.

Using Koszul's formula, we immediately obtain the following for a vector field ξ on M:

$$2g(\nabla_X \xi, Y) = (\pounds_\xi g)(X,Y) + d\eta(X,Y), \quad X,Y \in \mathfrak{X}(M),$$

where η is the 1-form dual to ξ, that is, $\eta(X) = g(X,\xi)$, $X \in \mathfrak{X}(M)$. Define a skew-symmetric tensor field φ of type $(1,1)$ on M by $d\eta(X,Y) = 2g(\varphi X, Y)$, and a symmetric tensor filed C of type $(1,1)$ by

$$\pounds_\xi g(X,Y) = 2g(CX,Y), \quad X,Y \in \mathfrak{X}(M),$$

then, for a smooth vector field ξ on M, we have

$$\nabla_X \xi = CX + \varphi X, \quad X,Y \in \mathfrak{X}(M). \tag{2.6}$$

Using the definition of a conformal vector field and equation (2.6), we have

Lemma 2.4 ([DES12]). *Let ξ be a conformal vector field on an n-dimensional Riemannian manifold $(M.g)$, with potential function ρ. Then*

$$\nabla_X \xi = \rho X + \varphi X, \quad X \in \mathfrak{X}(M) \quad \text{and} \quad \operatorname{div} \xi = n\rho.$$

Remark 2.1 ([DES08]). Let ξ be a conformal gradient vector field on a compact Riemannian manifold (M, g). Then, for $\rho = n^{-1} \operatorname{div} \xi$,

$$\int_M \rho dv = 0.$$

Let λ_1 be the nonzero eigenvalue of the Laplacian operator Δ acting on the smooth functions of a compact Riemannian manifold (M, g), where we adopt the sign convention of the Laplacian operator as $\Delta f = \operatorname{div} \nabla f$. Then, for a smooth function f on M satisfying

$$\int_M f dv = 0,$$

by minimum principle we have

$$\int_M \|\nabla f\|^2 dv \geq \lambda_1 \int_M f^2 dv, \tag{2.7}$$

and the equality holds if and only if $\Delta f = -\lambda_1 f$. Moreover, for a smooth function f, the Hessian operator H_f is given by

$$H_f X = \nabla_X \nabla f, \quad X \in \mathfrak{X}(M),$$

and on a compact Riemannian manifold, we have the following Bochner formula:

$$\int_M \left\{ \operatorname{Ric}(\nabla f, \nabla f) + \|H_f\|^2 - (\Delta f)^2 \right\} dv = 0. \tag{2.8}$$

3. Submanifolds with ψ^T as conformal vector field

Let M be an n-dimensional submanifold of the Euclidean space R^{n+p}, with immersion $\psi : M \to R^{n+p}$. In this section, we study the geometry of the submanifold M for which the vector field ψ^T is a conformal vector field. First, we prove the following Lemmas.

Lemma 3.1. *Let M be an n-dimensional submanifold of the Euclidean space R^{n+p}, with immersion $\psi : M \to R^{n+p}$ and $f = \frac{1}{2}\left\|\psi^{\perp}\right\|^{2}$. If the gradient ∇f of the smooth function f is a conformal vector field, then*

$$\operatorname{Ric}\left(\psi^{T}, \psi^{T}\right) + n\psi^{T}(F) + n\rho + nF + \|B\|^{2} = 0,$$

where ρ is the potential function of ∇f.

Proof. As ∇f is a conformal vector field with potential function say ρ, we have

$$\pounds_{\nabla f} g = 2\rho g.$$

Since the 1-form dual to the conformal vector field ∇f is closed, we have $\varphi = 0$, and Lemma 2.4 takes the form

$$\nabla_X(\nabla f) = \rho X \quad \text{and} \quad \Delta f = n\rho, \tag{3.1}$$

where Δ is the Laplacian operator. Now, for $X \in \mathfrak{X}(M)$, we have

$$\begin{aligned} g(\nabla f, X) = X(f) &= X\left(\frac{1}{2}\left\|\psi^{\perp}\right\|^{2}\right) = g\left(\overline{\nabla}_X \psi^{\perp}, \psi^{\perp}\right) = g\left(-A_{\psi^{\perp}}X + \nabla^{\perp}_X \psi^{\perp}, \psi^{\perp}\right) \\ &= g\left(\nabla^{\perp}_X \psi^{\perp}, \psi^{\perp}\right) = -g\left(h\left(X, \psi^{T}\right), \psi^{\perp}\right) = -g\left(A_{\psi^{\perp}}\psi^{T}, X\right), \end{aligned}$$

which gives $\nabla f = -A_{\psi^{\perp}}\psi^{T} = -B\psi^{T}$. Putting $\xi = \psi^{T}$, we get $\nabla f = -B\xi$, and consequently,

$$\nabla_X(\nabla f) = -\nabla_X B\xi = -\left[(\nabla B)(X, \xi) + B\nabla_X \xi\right],$$

which, using equation (2.4), gives

$$\begin{aligned} \nabla_X(\nabla f) &= -(\nabla B)(X, \xi) - B(X + BX) \\ &= -(\nabla B)(X, \xi) - BX - B^{2}X. \end{aligned} \tag{3.2}$$

Now, using Lemma 2.2 and the fact that B is a symmetric operator, we have

$$\begin{aligned} \sum_{i=1}^{n} g\left((\nabla B)(e_i, \xi), e_i\right) &= g\left(\sum_{i=1}^{n}(\nabla B)(e_i, e_i), \xi\right) \\ &= g\left(n\nabla F + Q(\xi), \xi\right) = n\xi(F) + \operatorname{Ric}(\xi, \xi). \end{aligned} \tag{3.3}$$

Also, using equations (3.1) and (3.2), we get

$$\begin{aligned} \sum_{i=1}^{n} g\left((\nabla B)(e_i, \xi), e_i\right) &= \sum_{i=1}^{n} g\left(-\rho e_i - Be_i - B^{2}e_i, e_i\right) \\ &= -n\rho - \operatorname{Tr} B - \|B\|^{2}. \end{aligned} \tag{3.4}$$

Then, using $\operatorname{Tr} B = nF$ and equations (3.3) and (3.4), we arrive at

$$\operatorname{Ric}(\xi, \xi) + n\xi(F) + n\rho + nF + \|B\|^2 = 0,$$

which proves the Lemma. □

Lemma 3.2. *Let $\psi : M \to R^{n+p}$ be an n-dimensional compact submanifold. Then*

$$\int_M \left\{ \operatorname{Ric}\left(\psi^T, \psi^T\right) - n^2 (1+F)^2 + \|B\|^2 - n \right\} dv = 0.$$

Proof. Taking $\xi = \psi^T$, we have

$$\operatorname{div}(F\xi) = g(\nabla F, \xi) + F \operatorname{div} \xi = g(\nabla F, \xi) + nF(1+F).$$

Consider a local orthonormal frame $\{e_1, ..., e_n\}$, then using Lemma 2.2 and equation (2.5) to compute $\operatorname{div}(B\xi)$, we get

$$\begin{aligned}
\operatorname{div}(B\xi) &= \sum_{i=1}^n g(\nabla_{e_i} B\xi, e_i) = \sum_{i=1}^n g((\nabla B)(e_i, \xi) + B\nabla_{e_i}\xi, e_i) \\
&= \sum_{i=1}^n [g((\nabla B)(e_i, e_i), \xi) + g(\nabla_{e_i}\xi, Be_i)] \\
&= g(n\nabla F + Q(\xi), \xi) + \sum_{i=1}^n [g(e_i, Be_i) + g(Be_i, Be_i)] \\
&= ng(\nabla F, \xi) + \operatorname{Ric}(\xi, \xi) + \operatorname{Tr} B + \|B\|^2 \\
&= ng(\nabla F, \xi) + \operatorname{Ric}(\xi, \xi) + nF + \|B\|^2,
\end{aligned}$$

and

$$g(\nabla F, \xi) = \operatorname{div}(F\xi) - nF^2 - nF,$$

which gives

$$ng(\nabla F, \xi) = n \operatorname{div}(F\xi) - n^2F^2 - n^2F.$$

Consequently,

$$\operatorname{div}(B\xi) = n \operatorname{div}(F\xi) - n^2F^2 - n^2F + \operatorname{Ric}(\xi, \xi) + nF + \|B\|^2,$$

and we have

$$\operatorname{div}(B\xi - nF\xi) = \operatorname{Ric}(\xi, \xi) - n^2F^2 - n^2F + nF + \|B\|^2,$$

which after integration gives

$$\int_M \left\{ \operatorname{Ric}(\xi,\xi) - n^2\left(F^2-1\right) + \|B\|^2 - n \right\} dv = 0. \tag{3.5}$$

Also using Lemma 2.1, we have

$$\int_M (1+F)^2\, dv = \int_M \left(F^2-1\right) dv,$$

which, together with equation (3.5), gives

$$\int_M \left\{ \operatorname{Ric}(\xi,\xi) - n^2 (1+F)^2 + \|B\|^2 - n \right\} dv = 0. \qquad \square$$

Theorem 3.1. *Let $\psi : M \to R^{n+p}$ be an n-dimensional compact submanifold with the tangential component ψ^T, a nonzero conformal vector field with potential function ρ, and λ_1 be the first nonzero eigenvalue of the Laplacian operator on the submanifold M. If $c = n^{-1}\lambda_1$ and the Ricci tensor on M satisfies*

(i) $\operatorname{Ric}\left(\nabla\rho + c\psi^T, \nabla\rho + c\psi^T\right) \geq 0$,

(ii) $\operatorname{Ric}(\nabla\rho,\nabla\rho) \leq (n-1)\, c\, \|\nabla\rho\|^2$,

then M is isometric to a sphere $S^n(c)$.

Proof. Let $\xi = \psi^T$ be a conformal vector field with potential function ρ. If we define $f = \frac{1}{2}\|\psi\|^2$, then it is easy to show that $\xi = \nabla f$. Thus ξ is a gradient conformal vector field, and consequently, as the 1-form η dual to ξ being $\eta = df$ is closed, we get that $\varphi = 0$. Then, by Lemma 2.4, we have

$$\nabla_X \xi = \rho X,$$

and using equation (2.4) in the above equation, we have

$$BX + X = \rho X,$$

which gives $B = (\rho - 1)\, I$ and $\operatorname{div}\xi = n\rho$. However, as $\xi = \nabla f$, we have $\Delta f = n\rho$.

Now,

$$(\nabla B)(X,Y) = \nabla_X BY - B\nabla_X Y = \nabla_X (\rho-1)\, Y - (\rho-1)\nabla_X Y = X(\rho)\, Y,$$

which, together with Lemma 2.2, gives

$$X(\rho)\, Y - Y(\rho)\, X = R(X,Y)\,\xi. \tag{3.6}$$

The above equation immediately gives

$$\operatorname{Ric}(\xi, X) = \sum_{i=1}^{n} R(e_i, X; \xi, e_i) = g(X, \nabla\rho) - nX(\rho),$$

and consequently, we have

$$Q(\xi) = -(n-1)\nabla\rho. \tag{3.7}$$

The above equation gives

$$\operatorname{Ric}(\xi, \xi) = -(n-1)\,\xi(\rho) = -(n-1)\left[\operatorname{div}(\rho\xi) - \rho \operatorname{div}\xi\right],$$

that is,

$$\operatorname{Ric}(\xi, \xi) = -(n-1)\operatorname{div}(\rho\xi) + n(n-1)\rho^2. \tag{3.8}$$

Also, equation (3.7) gives

$$\operatorname{Ric}(\xi, \nabla\rho) = g(-(n-1)\nabla\rho, \nabla\rho) = -(n-1)\left\|\nabla\rho\right\|^2. \tag{3.9}$$

Let λ_1 be the first nonzero eigenvalue of the Laplacian operator on M. Then Remark 2.1, together with equation (2.7), gives

$$\int_M \left\|\nabla\rho\right\|^2 dv \geq \lambda_1 \int_M \rho^2 dv, \tag{3.10}$$

with equality holding if and only if $\Delta\rho = -\lambda_1\rho$.

Using $c = n^{-1}\lambda_1$ and equations (3.8), (3.9) and (3.10), we arrive at

$$\begin{aligned} &\int_M \operatorname{Ric}(\nabla\rho + c\xi, \nabla\rho + c\xi)\, dv \\ &\quad= \int_M \left\{\operatorname{Ric}(\nabla\rho, \nabla\rho) + n(n-1)c^2\rho^2 - 2(n-1)c\left\|\nabla\rho\right\|^2\right\} dv \\ &\quad\leq \int_M \left\{\operatorname{Ric}(\nabla\rho, \nabla\rho) - (n-1)c\left\|\nabla\rho\right\|^2\right\} dv, \end{aligned}$$

Using the conditions in the statement, and the above inequality, we conclude that

$$\operatorname{Ric}(\nabla\rho + c\xi, \nabla\rho + c\xi) = 0 \quad \text{and} \quad \operatorname{Ric}(\nabla\rho, \nabla\rho) - (n-1)c\left\|\nabla\rho\right\|^2 = 0. \tag{3.11}$$

Thus we have

$$\operatorname{Ric}(\nabla\rho,\nabla\rho)+2c\operatorname{Ric}(\nabla\rho,\xi)+c^2\operatorname{Ric}(\xi,\xi)=0,$$

which, together with equation (3.9) and the second equation in (3.11), gives

$$\operatorname{Ric}(\xi,\xi)=(n-1)c^{-1}\left\|\nabla\rho\right\|^2. \tag{3.12}$$

Now, using $\nabla f=\xi$, that is, $H_f(X)=\rho X$ and $\Delta f=n\rho$ in the Bochner Formula (2.8), we arrive at

$$\int\limits_M \left\{\operatorname{Ric}(\xi,\xi)+n\rho^2-n^2\rho^2\right\}dv=0,$$

which, together with equation (3.12), gives

$$\int\limits_M \left\|\nabla\rho\right\|^2 dv = nc\int\limits_M \rho^2 dv=\lambda_1\int\limits_M \rho^2 dv.$$

This equality in (3.10) gives $\Delta\rho=-\lambda_1\rho$, which, together with $\Delta f=n\rho$, gives $\Delta(\rho+\lambda_1 n^{-1}f)=0$, and on compact M, we have $\rho+\lambda_1 n^{-1}f=\text{constant}$. This last equation, together with $H_f(X)=\rho X$, gives $\nabla\rho=-c\nabla f$, that is,

$$\nabla_X\nabla\rho=-c\rho X. \tag{3.13}$$

If ρ is a constant, then we have $-c\nabla f=0$, that is, $\xi=0$, which is a contradiction, as ξ is a nonzero conformal vector field. Hence the nonconstant function ρ satisfies the OBATA's differential equation (3.13) (cf. [OBA62]), and therefore is isometric to the sphere $S^n(c)$. □

In the following result, we consider the tangential component ψ^T and find conditions under which it becomes a conformal vector field on the submanifold M.

Theorem 3.2. *Let $\psi : M\to R^{n+p}$ be an n-dimensional compact submanifold, with $\lambda=\inf\frac{1}{n-1}\operatorname{Ric}>0$. If $\left\|\psi^T\right\|^2\geq n\lambda^{-1}(1+F)^2$, then ψ^T is a conformal vector field on M.*

PROOF. Taking $\xi=\psi^T$ in Lemma 3.2, we get

$$\int\limits_M \left\{\operatorname{Ric}(\xi,\xi)-n^2(1+F)^2+\left\|B\right\|^2-n\right\}dv=0,$$

which gives

$$\int\limits_M \left(\operatorname{Ric}(\xi,\xi)-\lambda(n-1)\|\xi\|^2\right)+(\|B\|^2-nF^2)+\left((n-1)\left(\lambda\|\xi\|^2-n(1+F)^2\right)\right)=0.$$

Using $\operatorname{Ric}(\xi,\xi) \geq (n-1)\lambda\|\xi\|^2$, the Schwarz inequality $\|B\|^2 \geq nF^2$ and the condition in the statement $\lambda\|\xi\|^2 \geq n(1+F)^2$ in the above equation, we get the equality $\|B\|^2 = nF^2$, which holds if and only if $B = FI$. Thus

$$\nabla_X \xi = BX + X = FX + X = (1+F)X = \rho X,$$

where $\rho = (1+F)$, that is,

$$\pounds_\xi g = 2\rho g,$$

which proves that $\xi = \psi^T$ is a conformal vector field. □

In the next result, we consider a conformal vector field on the submanifold M associated with the normal component $\psi^\perp$, and it is interesting to note that in this case we get the criterion for the submanifold to lie on the hypersphere in the Euclidean space, that is, we get a criterion for a spherical submanifold.

Theorem 3.3. *Let $\psi : M \to R^{n+p}$ be an n-dimensional compact submanifold with mean curvature H. Suppose that the smooth function $f = \frac{1}{2}\left\|\psi^\perp\right\|^2$ gives the conformal vector field ∇f on M, and that $\nabla^\perp_{\psi^T} H = 0$. Then $h\left(\psi^T,\psi^T\right) = 0$ if and only if $\psi(M) \subset S^{n+p-1}(c)$ for some constant $c > 0$.*

Proof. Suppose that $h\left(\psi^T,\psi^T\right) = 0$. Then, for $\xi = \psi^T$, we have

$$\xi(F) = g\left(\nabla^\perp_\xi H, \psi^\perp\right) + g\left(H, \nabla^\perp_\xi \psi^\perp\right) = -g(H, h(\xi,\xi)) = 0,$$

that is, $\xi(F) = 0$, which, together with Lemma 3.1, gives

$$\operatorname{Ric}(\xi,\xi) + n\xi(F) + n\rho + nF + \|B\|^2 = 0.$$

Integrating the above equation, we get

$$\int\limits_M \left\{\operatorname{Ric}(\xi,\xi) + nF + \|B\|^2\right\} dv = \int\limits_M \left\{\operatorname{Ric}(\xi,\xi) + \|B\|^2 - n\right\} dv = 0,$$

where we used Lemma 2.1.

Now, using Lemma 3.2 in the above equation, we get

$$\int\limits_M -n^2(1+F)^2\, dv = 0,$$

that is, $F = -1$, which, by virtue of Lemma 2.3, gives $\psi(M) \subset S^{n+p-1}(c)$ for some constant $c > 0$.

Conversely, if $\psi(M) \subset S^{n+p-1}(c)$, $c > 0$, then by Lemma 2.3 $F = -1$ and $\psi^T = 0$, and this proves $h(\xi,\xi) = 0$. □

4. Submanifolds with flat normal connection

In this section, we study codimension-two submanifolds in the Euclidean space R^{n+2} with flat normal connection, and find conditions under which the tangential component of the position vector field is a conformal vector field. Let $\psi : M \to R^{n+2}$ be an immersion of a compact manifold with a flat normal connection and a mean curvature vector field H. We assume that the mean curvature vector field H is nowhere zero, and choose a local orthonormal frame $\{N_1, N_2\}$ of normals such that $H = \alpha N_1$, where $\alpha = \|H\|$. Then, using the definition of the smooth function $F = \left\langle \psi^{\perp}, H \right\rangle$, in this case we have

$$\psi^{\perp} = \frac{F}{\alpha} N_1 + \mu N_2, \quad \mu = \left\langle N_2, \psi^{\perp} \right\rangle . \tag{4.1}$$

Define a smooth 1-form ω by $\omega(X) = g\left(\nabla^{\perp}_X N_1, N_2\right)$, $X \in \mathfrak{X}(M)$, and let υ be the smooth vector field on M dual to ω.

Lemma 4.1. *Let $\psi : M \to R^{n+2}$ be an immersion of a smooth manifold with a local orthonormal frame $\{N_1, N_2\}$ of normals such that $H = \alpha N_1$. Then, the normal connection on M is flat if and only if ω is closed.*

Proof. Using $\omega(X) = g\left(\nabla^{\perp}_X N_1, N_2\right)$, we have $\nabla^{\perp}_X N_1 = \omega(X) N_2$ and that $\nabla^{\perp}_X N_2 = -\omega(X) N_1$. We compute $R^{\perp}(X,Y) N_1$ to get

$$R^{\perp}(X,Y) N_1 = X(\omega(Y)) N_2 - Y(\omega(X)) N_2 - \omega([X,Y]) N_2 = d\omega(X,Y) N_2,$$

and similarly we have

$$R^{\perp}(X,Y) N_2 = -d\omega(X,Y) N_1, \quad X, Y \in \mathfrak{X}(M),$$

which proves the normal connection is flat if an only if $d\omega = 0$, that is, ω is closed. □

Let M be a submanifold of R^{n+2} with flat normal connection. Then as the smooth 1-form ω, which is dual to smooth vector field υ, is closed using equation (2.6), we have a symmetric tensor field C that is given by $\nabla_X \upsilon = CX$, for $X \in \mathfrak{X}(M)$.

Lemma 4.2. *Let $\psi : M \to R^{n+2}$ be an immersion of a smooth manifold with a local orthonormal frame $\{N_1, N_2\}$ of normals such that $H = \alpha N_1$ and shape operators $A_1 = A_{N_1}$ and $A_2 = A_{N_2}$. Then*

(i) $\sum_{i=1}^{n} (\nabla A_1)(e_i, e_i) = n\nabla\alpha + A_2 \upsilon$,

(ii) $\sum_{i=1}^{n} (\nabla A_2)(e_i, e_i) = n\alpha v - A_1 v,$

where $\{e_1, ..., e_n\}$ *is a local orthonormal frame on* M.

PROOF. Using the expression

$$(Dh)(X,Y,Z) = \nabla_X^{\perp} h(Y,Z) = h(\nabla_X Y, Z) - h(\nabla_X Z, Y),$$

and the Codazzi equation of the submanifold

$$(Dh)(X,Y,Z) = (Dh)(Y,Z,X), \quad X,Y,Z \in \mathfrak{X}(M),$$

we get

$$(\nabla A_1)(X,Y) - (\nabla A_1)(Y,X) = A_{\nabla_X^{\perp} N_1} Y - A_{\nabla_Y^{\perp} N_1} X, \tag{4.2}$$

and that

$$(\nabla A_2)(X,Y) - (\nabla A_2)(Y,X) = A_{\nabla_X^{\perp} N_2} Y - A_{\nabla_Y^{\perp} N_2} X. \tag{4.3}$$

Also we have

$$\operatorname{Tr} A_1 = n\alpha \quad \text{and} \quad \operatorname{Tr} A_2 = 0, \tag{4.4}$$

and consequently, we get

$$\sum_{i=1}^{n} g((\nabla A_1)(X,e_i), e_i) = \sum_{i=1}^{n} g(\nabla_X A_1 e_i, e_i) - g(A_1 \nabla_X e_i, e_i) = ng(X, \nabla\alpha).$$

Using equations (4.2) and (4.3) in the above equation, we arrive at the desired result in (i).

Similarly, using equations (4.3) and (4.4), we get

$$\sum_{i=1}^{n} g((\nabla A_2)(X,e_i), e_i) = \sum_{i=1}^{n} g(\nabla_X A_2 e_i, e_i) - g(A_2 \nabla_X e_i, e_i) = X(\operatorname{Tr} A_2) = 0,$$

and arrive at the desired result in (ii). □

In the following main result of this section, we find necessary conditions for the vector field $\xi = \psi^T$ on the submanifold M of the Euclidean space R^{n+2} with flat normal connection to be a conformal vector field. Let $\psi : M \to R^{n+2}$ be a compact submanifold with flat normal connection, and v be the vector field dual to the closed 1-form ω given in Lemma 4.1, and $h = \operatorname{Tr} C$, C being the symmetric tensor field given by $CX = \nabla_X v$.

Theorem 4.1. *Let $\psi : M \to R^{n+2}$ be an immersion of a compact manifold with a flat normal connection, and $\{N_1, N_2\}$ a local orthonormal frame of normals such that $H = \alpha N_1$, $H(p) \neq 0$, $p \in M$. If there is a constant c and the following conditions hold:*

(i) $\operatorname{Ric}(v, v) \geq \frac{n-1}{n} h^2$,

(ii) $\operatorname{Ric}(\xi - cv, \xi - cv) \geq 0$,

(iii) $|ch - nF| \leq n$,

where $\xi = \psi^T$, then ξ is a conformal vector field.

Proof. Using the definition of the curvature tensor field and

$$\nabla_X v = CX, \tag{4.5}$$

we get

$$R(X, Y)v = (\nabla C)(X, Y) - (\nabla C)(Y, X). \tag{4.6}$$

Since $h = \operatorname{Tr} C$, the above equation gives

$$\operatorname{Ric}(X.v) = g\left(\sum_{i=1}^{n} (\nabla C)(e_i, e_i) - \nabla h, X\right),$$

that is,

$$Q(v) = \sum_{i=1}^{n} (\nabla C)(e_i, e_i) - \nabla h. \tag{4.7}$$

Now, using equation (4.7) in computing $\operatorname{div} Cv$, we get

$$\operatorname{div} Cv = \operatorname{Ric}(v, v) + v(h) + \|C\|^2. \tag{4.8}$$

Also, equation (4.5) gives $\operatorname{div} v = h$, and thus we have

$$\operatorname{div} hv = v(h) + h^2,$$

which on integration gives

$$\int_M v(h)\, dv = -\int_M h^2 dv.$$

Now, integrating equation (4.8) and using the above equation, we get

$$\int_M \left\{\operatorname{Ric}(v, v) + \|C\|^2 - h^2\right\} dv = 0,$$

that is,

$$\int_M \left\{ \left(\operatorname{Ric}(v,v) - \frac{n-1}{n}h^2 \right) + \left(\|C\|^2 - \frac{1}{n}h^2 \right) \right\} dv = 0.$$

Thus the condition (i) in the statement, together with Schwarz inequality $\|C\|^2 \geq \frac{1}{n}h^2$, gives

$$\operatorname{Ric}(v,v) = \frac{n-1}{n}h^2 \quad \text{and} \quad \|C\|^2 = \frac{1}{n}h^2. \tag{4.9}$$

The second equation in (4.9) gives

$$C = \frac{h}{n}I \quad \text{and} \quad \nabla_X v = \frac{h}{n}X. \tag{4.10}$$

Now, using equation (4.7), we get

$$\operatorname{Ric}(v,v) = -\left(\frac{n-1}{n}\right) v(h),$$

which, together with equation (4.9), gives $v(h) = -h^2$. Also, the first equation in (4.10) and $\operatorname{Tr} B = F$ give $\operatorname{Tr} CB = hF$.

Using equation (4.1) in (2.4), we get

$$X\left(\frac{F}{\alpha}\right) N_1 + \frac{F}{\alpha}\nabla^{\perp}_X N_1 + X(\mu) N_2 + \mu \nabla^{\perp}_X N_2 = -h(X,\xi), \tag{4.11}$$

which, taking inner product with N_1, gives

$$\nabla\left(\frac{F}{\alpha}\right) = \mu v - A_1\xi, \tag{4.12}$$

similarly, taking inner product with N_2 gives

$$\nabla\mu = -A_2\xi - \frac{F}{\alpha}v. \tag{4.13}$$

Now, we compute the divergence of the vector field Bv,

$$\operatorname{div} Bv = \sum_{i=1}^{n} g\left(\nabla_{e_i} Bv, e_i\right) = \sum_{i=1}^{n} g\left(\nabla_{e_i}\left(\frac{F}{\alpha}A_1 v + \mu A_2 v\right), e_i\right),$$

which, using equations (4.4), (4.12), (4.13) and Lemma 4.1, gives

$$\operatorname{div} Bv = -g\left(A_1^2 v + A_2^2 v, \xi\right) + n\frac{F}{\alpha}v(\alpha) + n\alpha\mu\|v\|^2 + Fh. \tag{4.14}$$

On using equation (4.12), we get

$$v\left(F\right)=\frac{F}{\alpha}v\left(\alpha\right)+\alpha\mu\left\Vert v\right\Vert ^{2}-\alpha g\left(A_{1}v,\xi\right), \tag{4.15}$$

and

$$\operatorname{div}Fv=F\operatorname{div}v+v\left(F\right)=hF+\frac{F}{\alpha}v\left(\alpha\right)+\alpha\mu\left\Vert v\right\Vert ^{2}-\alpha g\left(A_{1}v,\xi\right),$$

and consequently, that

$$n\operatorname{div}Fv+n\alpha g\left(A_{1}v,\xi\right)=n\frac{F}{\alpha}v\left(\alpha\right)+n\alpha\mu\left\Vert v\right\Vert ^{2}+nhF. \tag{4.16}$$

Now, using the expression for the Ricci tensor of submanifold, we have

$$\operatorname{Ric}\left(X,v\right)=ng\left(h\left(v,X\right),H\right)-\sum_{i=1}^{n}g\left(h\left(X,e_{i}\right),h\left(v,e_{i}\right)\right),$$

which gives

$$Q\left(v\right)=n\alpha A_{1}v-A_{1}^{2}v-A_{2}^{2}v. \tag{4.17}$$

Using equations (4.16) and (4.17) in equation (4.14), we get

$$\operatorname{div}Bv=\operatorname{Ric}\left(\xi,v\right)+n\operatorname{div}Fv-\left(n-1\right)hF,$$

and integrating the above equation we have

$$\int_{M}\left\{\operatorname{Ric}\left(\xi,v\right)-\left(n-1\right)hF\right\}dv=0. \tag{4.18}$$

Finally, using equations (4.9) and (4.18) and Lemma 3.2, we get

$$\int_{M}\operatorname{Ric}\left(\xi-cv,\xi-cv\right)dv=\int_{M}\left\{\left(nF^{2}-\left\Vert B\right\Vert ^{2}\right)+\frac{n-1}{n}\left[\left(ch-nF\right)^{2}-n^{2}\right]\right\}dv,$$

which, together with the conditions in the statement and the Schwarz inequality $\left\Vert B\right\Vert^{2}\geq nF^{2}$, gives

$$\left\Vert B\right\Vert ^{2}=nF^{2},\quad \xi=cv\quad \text{and}\quad \left|ch-nF\right|=n.$$

The second equation, together with equation (4.10), gives

$$\nabla_{X}\xi=\frac{c}{n}hX,\quad X\in\mathfrak{X}\left(M\right).$$

This proves that

$$\left(\pounds_{\xi}g\right)\left(X,Y\right)=2\frac{c}{n}hg\left(X,Y\right),$$

that is, $\xi=\psi^{T}$ is a conformal vector field. □

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