4. The summation formula found in the example in Sec. 52 can be written

$$\sum_{n=1}^{\infty} z^n = \frac{z}{1-z} \quad \text{when} \quad |z|<1.$$

If we put $z = re^{i\theta}$, where $0<r<1$, the left-hand side becomes

$$\sum_{n=1}^{\infty} (re^{i\theta})^n = \sum_{n=1}^{\infty} r^n e^{in\theta} = \sum_{n=1}^{\infty} r^n \cos n\theta + i\sum_{n=1}^{\infty} r^n \sin n\theta;$$

and the right-hand side takes the form

$$\frac{re^{i\theta}}{1-re^{i\theta}} \cdot \frac{1-re^{-i\theta}}{1-re^{-i\theta}} = \frac{re^{i\theta}-r^2}{1-r(e^{i\theta}+e^{-i\theta})+r^2} = \frac{r\cos\theta - r^2 + ir\sin\theta}{1-2r\cos\theta+r^2}.$$

Thus

$$\sum_{n=1}^{\infty} r^n \cos n\theta + i\sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r\cos\theta - r^2}{1-2r\cos\theta+r^2} + i\frac{r\sin\theta}{1-2r\cos\theta+r^2}.$$

Equating the real parts on each side here and then the imaginary parts, we arrive at the summation formulas

$$\sum_{n=1}^{\infty} r^n \cos n\theta = \frac{r\cos\theta - r^2}{1-2r\cos\theta+r^2} \quad \text{and} \quad \sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r\sin\theta}{1-2r\cos\theta+r^2},$$

where $0<r<1$. These formulas clearly hold when $r=0$ too.

SECTION 54

1. Replace z by z^2 in the known series

$$\cosh z = \sum_{n-0}^{\infty} \frac{z^{2n}}{(2n)!} \qquad (|z| < \infty)$$

to get

$$\cosh(z^2) = \sum_{n=0}^{\infty} \frac{z^{4n}}{(2n)!} \qquad (|z| < \infty).$$

Then, multiplying through this last equation by z, we have the desired result:

$$z\cosh(z^2) = \sum_{n=0}^{\infty} \frac{z^{4n+1}}{(2n)!} \qquad (|z| < \infty).$$

2. *(b)* Replacing z by $z-1$ in the known expansion

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad (|z| < \infty),$$

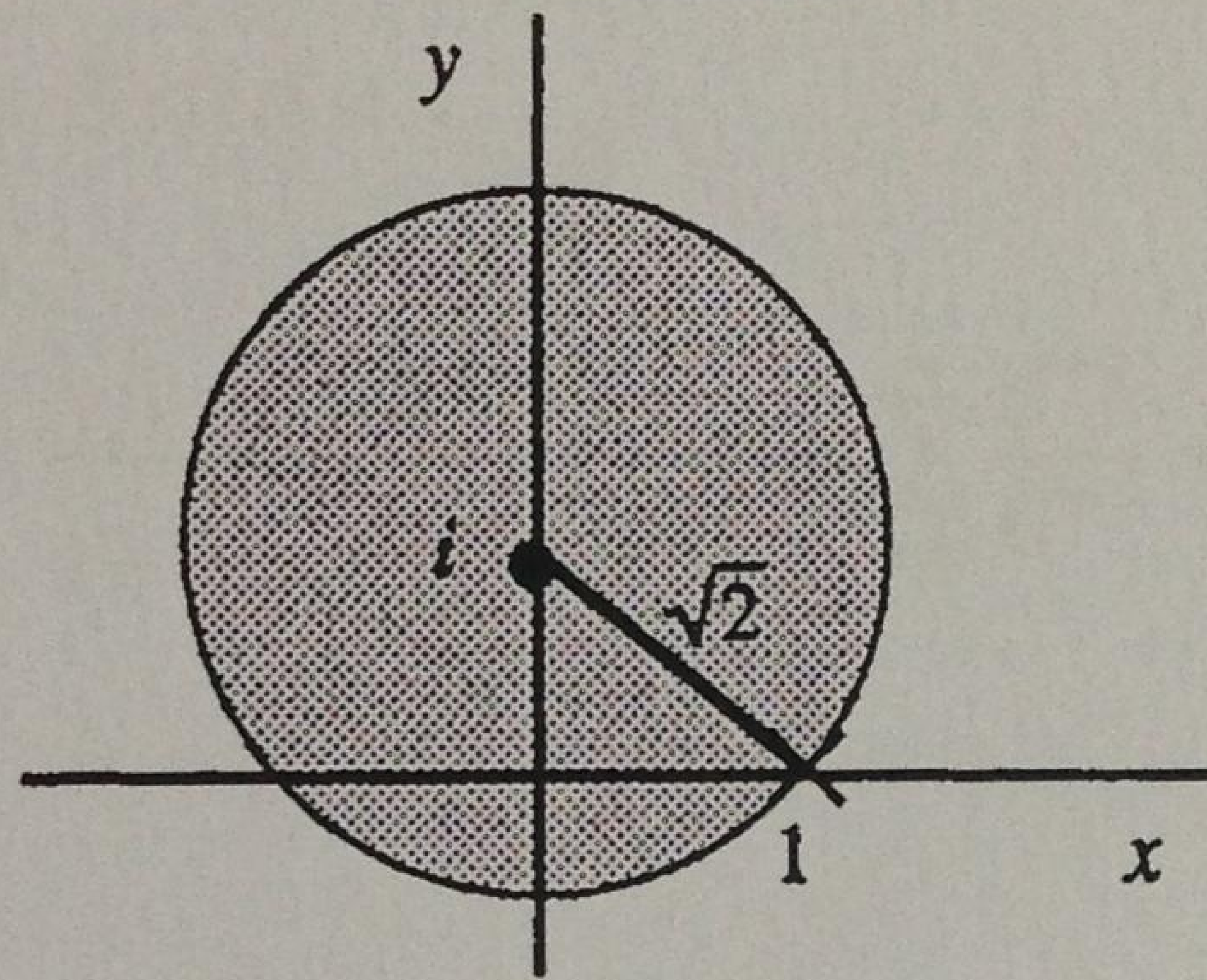
we have

$$e^{z-1} = \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \qquad (|z| < \infty).$$

So

$$e^z = e^{z-1}e = e\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \qquad (|z| < \infty).$$

7. The function $\frac{1}{1-z}$ has a singularity at $z = 1$. So the Taylor series about $z = i$ is valid when $|z - i| < \sqrt{2}$, as indicated in the figure below.



To find the series, we start by writing

$$\frac{1}{1-z} = \frac{1}{(1-i)-(z-i)} = \frac{1}{1-i} \cdot \frac{1}{1-(z-i)/(1-i)}.$$

This suggests that we replace z by $(z-i)/(1-i)$ in the known expansion

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (|z| < 1)$$

and then multiply through by $\frac{1}{1-i}$. The desired Taylor series is then obtained:

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}} \qquad (|z-i| < \sqrt{2}).$$

$$\cdots \frac{4z}{1-\frac{z}{4}} \cdots 4z \sum_{n=0} \left(\frac{z}{4}\right) \cdots = \sum_{n=0} \frac{\cdots}{4^{n+1}} = \frac{\cdots}{4z} + \sum_{n=1} \frac{z}{4^{n+1}} = \frac{1}{4\cdots}$$

SECTION 56

1. We may use the expansion

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

to see that when $0 < |z| < \infty$,

$$z^2 \sin\left(\frac{1}{z^2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{1}{z^{4n}} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{1}{z^{4n}}.$$

3. Suppose that $1 < |z| < \infty$ and recall the Maclaurin series representation

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

This enables us to write

$$\frac{1}{1+z} = \frac{1}{z} \cdot \frac{1}{1+\frac{1}{z}} = \frac{1}{z} \sum_{n=0}^{\infty} \left(-\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{n+1}}$$

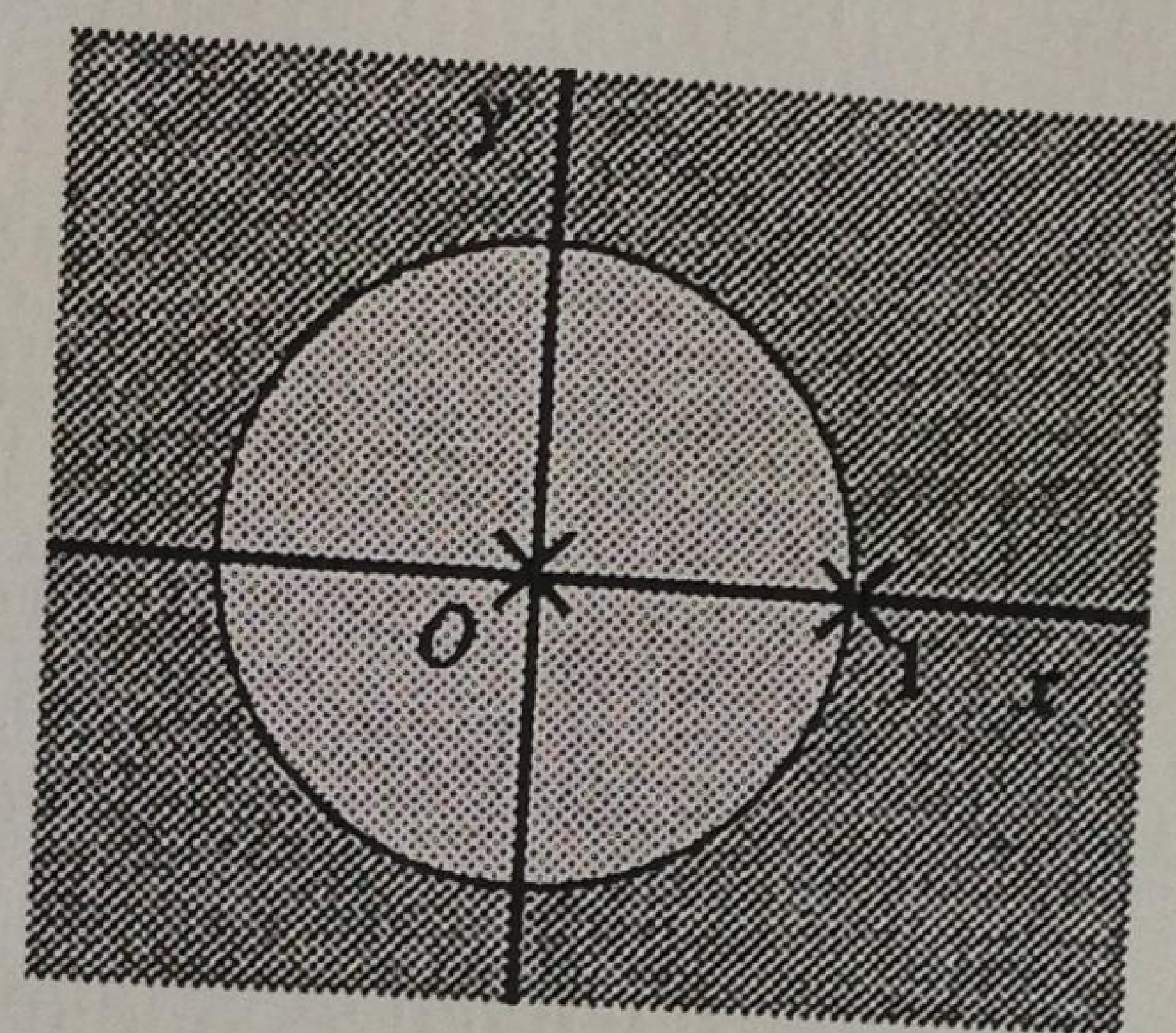
Replacing n by $n-1$ in this last series and then noting that

$$(-1)^{n-1} = (-1)^{n-1}(-1)^2 = (-1)^{n+1},$$

we arrive at the desired expansion:

$$\frac{1}{1+z}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^n} \qquad (1<|z|<\infty).$$

4. The singularities of the function $f(z)=\dfrac{1}{z^2(1-z)}$ are at the points $z=0$ and $z=1$. Hence there are Laurent series in powers of z for the domains $0<|z|<1$ and $1<|z|<\infty$ (see the figure below).



To find the series when $0<|z|<1$, recall that $\dfrac{1}{1-z}=\sum_{n=0}^{\infty}z^n$ $(|z|<1)$ and write

$$f(z)=\frac{1}{z^2}\cdot\frac{1}{1-z}=\frac{1}{z^2}\sum_{n=0}^{\infty}z^n=\sum_{n=0}^{\infty}z^{n-2}=\frac{1}{z^2}+\frac{1}{z}+\sum_{n=2}^{\infty}z^{n-2}=\sum_{n=0}^{\infty}z^n+\frac{1}{z}+\frac{1}{z^2}.$$

As for the domain $1<|z|<\infty$, note that $|1/z|<1$ and write

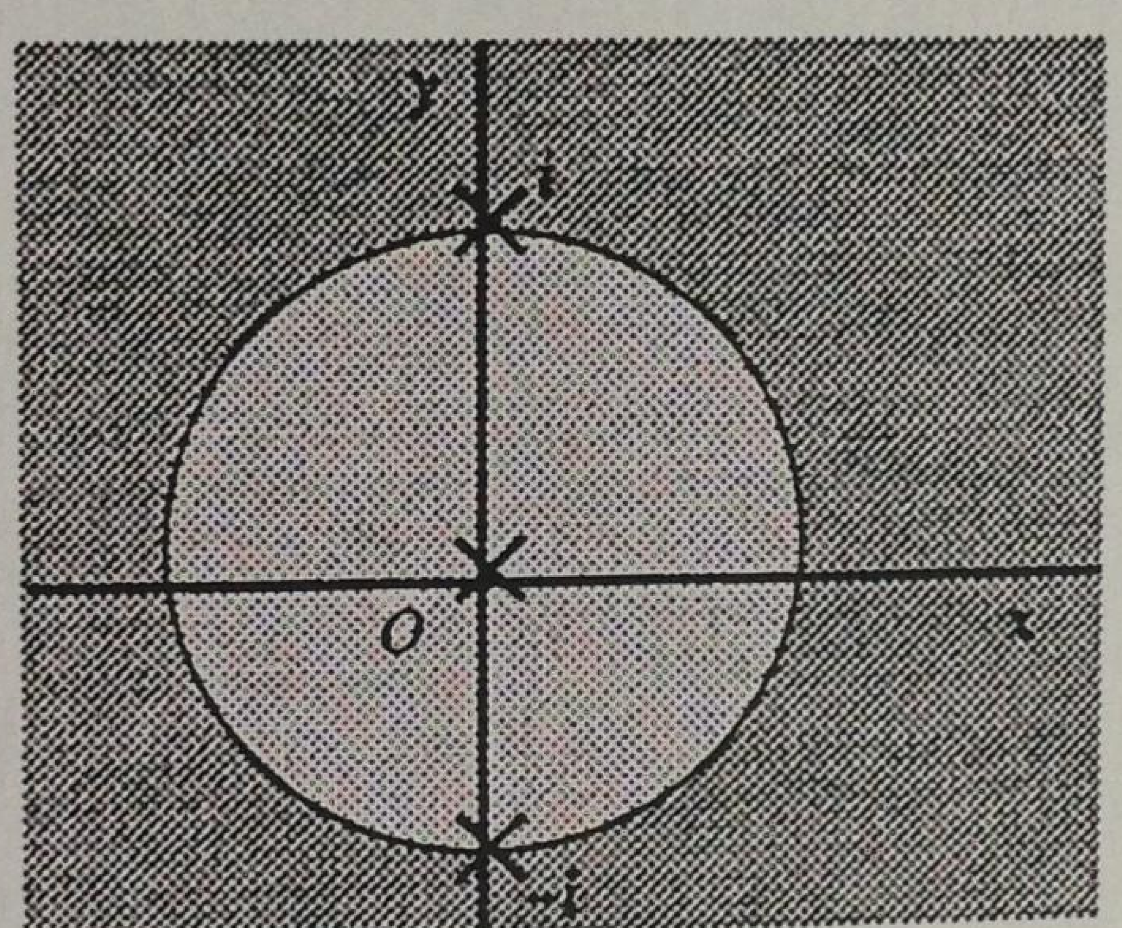
$$f(z)=-\frac{1}{z^3}\cdot\frac{1}{1-(1/z)}=-\frac{1}{z^3}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n=-\sum_{n=0}^{\infty}\frac{1}{z^{n+3}}=-\sum_{n=3}^{\infty}\frac{1}{z^n}.$$

(b) To find the Laurent series for the same function when $1<|z|<\infty$, we recall the Maclaurin series for $\frac{1}{1-z}$ that was used in part *(a)*. Since $\left|\frac{1}{z}\right|<1$ here, we may write

$$\frac{z+1}{z-1}=\frac{1+\frac{1}{z}}{1-\frac{1}{z}}=\left(1+\frac{1}{z}\right)\frac{1}{1-\frac{1}{z}}=\left(1+\frac{1}{z}\right)\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n=\sum_{n=0}^{\infty}\frac{1}{z^n}+\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$

$$=\sum_{n=0}^{\infty}\frac{1}{z^n}+\sum_{n=1}^{\infty}\frac{1}{z^n}=1+2\sum_{n=1}^{\infty}\frac{1}{z^n} \qquad (1<|z|<\infty).$$

7. The function $f(z)=\frac{1}{z(1+z^2)}$ has isolated singularities at $z=0$ and $z=\pm i$, as indicated in the figure below. Hence there is a Laurent series representation for the domain $0<|z|<1$ and also one for the domain $1<|z|<\infty$, which is exterior to the circle $|z|=1$.



To find each of these Laurent series, we recall the Maclaurin series representation

$$\frac{1}{1-z}=\sum_{n=0}^{\infty}z^n \qquad (|z|<1).$$

For the domain $0<|z|<1$, we have

$$f(z)=\frac{1}{z}\cdot\frac{1}{1+z^2}=\frac{1}{z}\sum_{n=0}^{\infty}(-z^2)^n=\sum_{n=0}^{\infty}(-1)^n z^{2n-1}=\frac{1}{z}+\sum_{n=1}^{\infty}(-1)^n z^{2n-1}=\sum_{n=0}^{\infty}(-1)^{n+1}z^{2n+1}+\frac{1}{z}.$$

On the other hand, when $1<|z|<\infty$,

$$f(z)=\frac{1}{z^3}\cdot\frac{1}{1+\frac{1}{z^2}}=\frac{1}{z^3}\sum_{n=0}^{\infty}\left(-\frac{1}{z^2}\right)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n+3}}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^{2n+1}}.$$

In this second expansion, we have used the fact that $(-1)^{n-1}=(-1)^{n-1}(-1)^2=(-1)^{n+1}$.

8. *(a)* Let a denote a real number, where $-1 < a < 1$. Recalling that

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (|z| < 1)$$

enables us to write

$$\frac{a}{z-a} = \frac{a}{z} \cdot \frac{1}{1-(a/z)} = \sum_{n=0}^{\infty} \frac{a^{n+1}}{z^{n+1}},$$

or

$$\frac{a}{z-a} = \sum_{n=1}^{\infty} \frac{a^n}{z^n} \qquad (|a| < |z| < \infty).$$

(b) Putting $z = e^{i\theta}$ on each side of the final result in part *(a)*, we have

$$\frac{a}{e^{i\theta} - a} = \sum_{n=1}^{\infty} a^n e^{-in\theta}.$$

But

$$\frac{a}{e^{i\theta} - a} = \frac{a}{(\cos\theta - a) + i\sin\theta} \cdot \frac{(\cos\theta - a) - i\sin\theta}{(\cos\theta - a) - i\sin\theta} = \frac{a\cos\theta - a^2 - ia\sin\theta}{1 - 2a\cos\theta + a^2}$$

and

$$\sum_{n=1}^{\infty} a^n e^{-in\theta} = \sum_{n=1}^{\infty} a^n \cos n\theta - i \sum_{n=1}^{\infty} a^n \sin n\theta.$$

Consequently,

$$\sum_{n=1}^{\infty} a^n \cos n\theta = \frac{a\cos\theta - a^2}{1 - 2a\cos\theta + a^2} \quad \text{and} \quad \sum_{n=1}^{\infty} a^n \sin n\theta = \frac{a\sin\theta}{1 - 2a\cos\theta + a^2}$$

when $-1 < a < 1$.

$$\sin\left(\frac{1}{z^2}\right) = z^2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{-4n-2} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\frac{1}{z^{4n}}$$

for $0 < |z| < \infty$ by substituting z^{-2} for z.

2, 30

sec. 58

180

(6) Derive the Laurent series representation

$$\frac{e^z}{(z+1)^2} = \frac{1}{e}\left(\sum_{n=0}^{\infty}\frac{(z+1)^n}{(n+2)!} + \frac{1}{z+1} + \frac{1}{(z+1)^2}\right) \quad \text{for } 0 < |z+1| < \infty$$

Solution. Since

$$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}$$

for $|z| < \infty$, we have

$$e^{z+1} = \sum_{n=0}^{\infty}\frac{z+1^n}{n!}$$

for $|z+1| < \infty$ by substituting $z+1$ for z. Therefore,

$$\frac{\times e}{\times e}\ \frac{e^z}{(z+1)^2} = \frac{e^{z+1}}{e(z+1)^2} = \sum_{n=0}^{\infty}\frac{(z+1)^{n-2}}{e(n!)}$$

$$= \frac{1}{e}\left(\frac{1}{(z+1)^2} + \frac{1}{z+1} + \sum_{n=2}^{\infty}\frac{(z+1)^{n-2}}{n!}\right)$$

$$= \frac{1}{e}\left(\frac{1}{(z+1)^2} + \frac{1}{z+1} + \sum_{n=0}^{\infty}\frac{(z+1)^n}{(n+2)!}\right)$$

for $0 < |z+1| < \infty$.

4, 30

sec. 58

198

(7) Give two Laurent Series expansions in powers of z for the function

$$f(z) = \frac{1}{z^2(1-z)}$$

and specify the regions in which those expansions are valid.

س 6
sec. 56
ص ١٩٨

(8) Show that when $0 < |z-1| < 2$,

$$\frac{z}{(z-1)(z-3)} = -3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}} - \frac{1}{2(z-1)}.$$

Solution. We first write $z/((z-1)(z-3))$ as a sum of partial fractions:

$$\frac{z}{(z-1)(z-3)} = \frac{1}{2}\left(-\frac{1}{z-1} + \frac{3}{z-3}\right)$$

When $0 < |z-1| < 2$, $|(z-1)/2| < 1$ and hence

$$\frac{1}{z-3} = -\frac{1}{2-(z-1)} = -\frac{1}{2}\frac{1}{1-(z-1)/2}$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^n} = -\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+1}}$$

Therefore,

$$\frac{z}{(z-1)(z-3)} = \frac{1}{2}\left(-\frac{1}{z-1} + \frac{3}{z-3}\right)$$

$$= -\frac{3}{2}\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+1}} - \frac{1}{2(z-1)}$$

$$= -3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}} - \frac{1}{2(z-1)}$$

SECTION 60

1. Differentiating each side of the representation

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (|z|<1),$$

we find that

$$\frac{1}{(1-z)^2} = \frac{d}{dz}\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} \frac{d}{dz} z^n = \sum_{n=1}^{\infty} nz^{n-1} = \sum_{n=0}^{\infty} (n+1)z^n \qquad (|z|<1).$$

Another differentiation gives

$$\frac{2}{(1-z)^3} = \frac{d}{dz}\sum_{n=0}^{\infty} (n+1)z^n = \sum_{n=0}^{\infty} (n+1)\frac{d}{dz} z^n = \sum_{n=1}^{\infty} n(n+1)z^{n-1} = \sum_{n=0}^{\infty} (n+1)(n+2)z^n \qquad (|z|<1).$$

2. Replace z by $1/(1-z)$ on each side of the Maclaurin series representation (Exercise 1)

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1)z^n \qquad (|z|<1),$$

as well as in its condition of validity. This yields the Laurent series representation

$$\frac{1}{z^2} = \sum_{n=2}^{\infty} \frac{(-1)^n(n-1)}{(z-1)^n} \qquad (1<|z-1|<\infty).$$

SECTION 64

1. *(a)* Let us write

C.71

$$\frac{1}{z+z^2}=\frac{1}{z}\cdot\frac{1}{1+z}=\frac{1}{z}\left(1-z+z^2-z^3+\cdots\right)=\frac{1}{z}-1+z-z^2+\cdots \qquad (0<|z|<1)$$

The residue at $z=0$, which is the coefficient of $\frac{1}{z}$, is clearly 1.

(b) We may use the expansion

$$\cos z = 1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots \qquad (|z|<\infty)$$

to write

$$z\cos\left(\frac{1}{z}\right)=z\left(1-\frac{1}{2!}\cdot\frac{1}{z^2}+\frac{1}{4!}\cdot\frac{1}{z^4}-\frac{1}{6!}\cdot\frac{1}{z^6}+\cdots\right)=z-\frac{1}{2!}\cdot\frac{1}{z}+\frac{1}{4!}\cdot\frac{1}{z^3}-\frac{1}{6!}\cdot\frac{1}{z^5}+\cdots$$

$$(0<|z|<\infty).$$

The residue at $z=0$, or coefficient of $\frac{1}{z}$, is now seen to be $-\frac{1}{2}$.

(c) Observe that

$$\frac{z-\sin z}{z}=\frac{1}{z}(z-\sin z)=\frac{1}{z}\left[z-\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right)\right]=\frac{z^2}{3!}-\frac{z^4}{5!}+\cdots \qquad (0<|z|<\infty).$$

Since the coefficient of $\frac{1}{z}$ in this Laurent series is 0, the residue at $z=0$ is 0.

(d) Write

$$\frac{\cot z}{z^4}=\frac{1}{z^4}\cdot\frac{\cos z}{\sin z}$$

and recall that

$$\cos z = 1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots=1-\frac{z^2}{2}+\frac{z^4}{24}-\cdots \qquad (|z|<\infty)$$

and

$$\sin z = z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots=z-\frac{z^3}{6}+\frac{z^5}{120}-\cdots \qquad (|z|<\infty).$$

Sol. ... 71

Dividing the series for $\sin z$ into the one for $\cos z$, we find that

$$\frac{\cos z}{\sin z} = \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} + \cdots \qquad (0 < |z| < \pi).$$

Thus

$$\frac{\cot z}{z^4} = \frac{1}{z^4}\left(\frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} + \cdots\right) = \frac{1}{z^5} - \frac{1}{3}\cdot\frac{1}{z^3} - \frac{1}{45}\cdot\frac{1}{z} + \cdots \qquad (0 < |z| < \pi).$$

Note that the condition of validity for this series is due to the fact that $\sin z = 0$ when $z = n\pi\ (n = 0, \pm 1, \pm 2, \ldots)$. It is now evident that $\dfrac{\cot z}{z^4}$ has residue $-\dfrac{1}{45}$ at $z = 0$.

(*e*) Recall that

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots \qquad (|z| < \infty)$$

and

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots \qquad (|z| < \infty).$$

There is a Laurent series for the function

$$\frac{\sinh z}{z^4(1-z^2)} = \frac{1}{z^4}\cdot(\sinh z)\left(\frac{1}{1-z^2}\right)$$

that is valid for $0 < |z| < 1$. To find it, we first multiply the Maclaurin series for $\sinh z$ and $\dfrac{1}{1-z^2}$:

$$\begin{aligned}
(\sinh z)\left(\frac{1}{1-z^2}\right) &= \left(z + \frac{1}{6}z^3 + \frac{1}{120}z^5 + \cdots\right)(1 + z^2 + z^4 + \cdots) \\
&= z + \frac{1}{6}z^3 + \frac{1}{120}z^5 + \cdots \\
&\qquad\; z^3 + \frac{1}{6}z^5 + \cdots \\
&\qquad\qquad\quad z^5 + \cdots \qquad\qquad (0 < |z| < 1). \\
&= z + \frac{7}{6}z^3 + \cdots
\end{aligned}$$

We then see that

$$\frac{\sinh z}{z^4(1-z^2)} = \frac{1}{z^3} + \frac{7}{6}\cdot\frac{1}{z} + \cdots \qquad (0 < |z| < 1).$$

This shows that the residue of $\dfrac{\sinh z}{z^4(1-z^2)}$ at $z = 0$ is $\dfrac{7}{6}$.

2. In each part, C denotes the positively oriented circle $|z| = 3$.

حل
2س
239ص
sec. 71

(a) To evaluate $\int_C \dfrac{\exp(-z)}{z^2}\,dz$, we need the residue of the integrand at $z = 0$. From the Laurent series

$$\frac{\exp(-z)}{z^2} = \frac{1}{z^2}\left(1 - \frac{z}{1!} + \frac{z^2}{2!} - \frac{z^3}{3!} + \cdots\right) = \frac{1}{z^2} - \frac{1}{1!}\cdot\frac{1}{z} + \frac{1}{2!} - \frac{z}{3!} + \cdots \qquad (0 < |z| < \infty),$$

we see that the required residue is -1. Thus

$$\int_C \frac{\exp(-z)}{z^2}\,dz = 2\pi i(-1) = -2\pi i.$$

(c) Likewise, to evaluate the integral $\int_C z^2 \exp\left(\dfrac{1}{z}\right) dz$, we must find the residue of the integrand at $z = 0$. The Laurent series

$$z^2 \exp\left(\frac{1}{z}\right) = z^2\left(1 + \frac{1}{1!}\cdot\frac{1}{z} + \frac{1}{2!}\cdot\frac{1}{z^2} + \frac{1}{3!}\cdot\frac{1}{z^3} + \frac{1}{4!}\cdot\frac{1}{z^4} + \cdots\right)$$

$$= z^2 + \frac{z}{1!} + \frac{1}{2!} + \frac{1}{3!}\cdot\frac{1}{z} + \frac{1}{4!}\cdot\frac{1}{z^2} + \cdots,$$

which is valid for $0 < |z| < \infty$, tells us that the needed residue is $\dfrac{1}{6}$. Hence

$$\int_C z^2 \exp\left(\frac{1}{z}\right) dz = 2\pi i\left(\frac{1}{6}\right) = \frac{\pi i}{3}.$$

2

239

sec.71

(*d*) As for the integral $\int_C \frac{z+1}{z^2-2z}\,dz$, we need the two residues of

$$\frac{z+1}{z^2-2z}=\frac{z+1}{z(z-2)},$$

one at $z=0$ and one at $z=2$. The residue at $z=0$ can be found by writing

$$\frac{z+1}{z(z-2)}=\left(\frac{z+1}{z}\right)\left(\frac{1}{z-2}\right)=\left(-\frac{1}{2}\right)\left(1+\frac{1}{z}\right)\cdot\frac{1}{1-(z/2)}$$

$$=\left(-\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{z}\right)\left(1+\frac{z}{2}+\frac{z^2}{2^2}+\cdots\right),$$

which is valid when $0<|z|<2$, and observing that the coefficient of $\frac{1}{z}$ in this last product is $-\frac{1}{2}$. To obtain the residue at $z=2$, we write

$$\frac{z+1}{z(z-2)}=\frac{(z-2)+3}{z-2}\cdot\frac{1}{2+(z-2)}=\frac{1}{2}\left(1+\frac{3}{z-2}\right)\cdot\frac{1}{1+(z-2)/2}$$

$$=\frac{1}{2}\left(1+\frac{3}{z-2}\right)\left[1-\frac{z-2}{2}+\frac{(z-2)^2}{2^2}-\cdots\right],$$

which is valid when $0<|z-2|<2$, and note that the coefficient of $\frac{1}{z-2}$ in this product is $\frac{3}{2}$. Finally, then, by the residue theorem,

$$\int_C \frac{z+1}{z^2-2z}\,dz=2\pi i\left(-\frac{1}{2}+\frac{3}{2}\right)=2\pi i.$$

3. In each part of this problem, C is the positively oriented circle $|z|=2$.

3

239

sec.71

(*a*) If $f(z)=\frac{z^5}{1-z^3}$, then

$$\frac{1}{z^2}f\left(\frac{1}{z}\right)=\frac{1}{z^7-z^4}=-\frac{1}{z^4}\cdot\frac{1}{1-z^3}=-\frac{1}{z^4}(1+z^3+z^6+\cdots)=-\frac{1}{z^4}-\frac{1}{z}-z^2-\cdots$$

when $0<|z|<1$. This tells us that

$$\int_C f(z)\,dz=2\pi i \operatorname*{Res}_{z=0}\frac{1}{z^2}f\left(\frac{1}{z}\right)=2\pi i(-1)=-2\pi i.$$

ت ٢٠
س 3
ص 239
sec. 71

(b) When $f(z)=\dfrac{1}{1+z^2}$, we have

$$\frac{1}{z^2}f\left(\frac{1}{z}\right)=\frac{1}{1+z^2}=\frac{1}{1-(-z^2)}=1-z^2+z^4-\cdots \qquad (0<|z|<1).$$

Thus

$$\int_C f(z)\,dz=2\pi i \operatorname*{Res}_{z=0}\frac{1}{z^2}f\left(\frac{1}{z}\right)=2\pi i(0)=0.$$

(c) If $f(z)=\dfrac{1}{z}$, it follows that $\dfrac{1}{z^2}f\left(\dfrac{1}{z}\right)=\dfrac{1}{z}$. Evidently, then,

$$\int_C f(z)\,dz=2\pi i \operatorname*{Res}_{z=0}\frac{1}{z^2}f\left(\frac{1}{z}\right)=2\pi i(1)=2\pi i.$$

SECTION 65

1. (a) From the expansion

243 ص

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

we see that

sec. 72

$$z\exp\left(\frac{1}{z}\right) = z + 1 + \frac{1}{2!}\cdot\frac{1}{z} + \frac{1}{3!}\cdot\frac{1}{z^2} + \cdots$$

The principal part of $z\exp\left(\frac{1}{z}\right)$ at the isolated singular point $z=0$ is, then,

$$\frac{1}{2!}\cdot\frac{1}{z}+\frac{1}{3!}\cdot\frac{1}{z^2}+\cdots;$$

C.72 and $z=0$ is an essential singular point of that function.

(b) The isolated singular point of $\dfrac{z^2}{1+z}$ is at $z=-1$. Since the principal part at $z=-1$ involves powers of $z+1$, we begin by observing that

$$z^2=(z+1)^2-2z-1=(z+1)^2-2(z+1)+1.$$

This enables us to write

$$\frac{z^2}{1+z}=\frac{(z+1)^2-2(z+1)+1}{z+1}=(z+1)-2+\frac{1}{z+1}.$$

Since the principal part is $\dfrac{1}{z+1}$, the point $z=-1$ is a (simple) pole.

(c) The point $z=0$ is the isolated singular point of $\dfrac{\sin z}{z}$, and we can write

$$\frac{\sin z}{z}=\frac{1}{z}\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right)=1-\frac{z^2}{3!}+\frac{z^4}{5!}-\cdots \qquad (0<|z|<\infty).$$

The principal part here is evidently 0, and so $z=0$ is a removable singular point of the function $\dfrac{\sin z}{z}$.

(d) The isolated singular point of $\dfrac{\cos z}{z}$ is $z=0$. Since

$$\frac{\cos z}{z}=\frac{1}{z}\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots\right)=\frac{1}{z}-\frac{z}{2!}+\frac{z^3}{4!}-\cdots \qquad (0<|z|<\infty),$$

the principal part is $\dfrac{1}{z}$. This means that $z=0$ is a (simple) pole of $\dfrac{\cos z}{z}$.

(e) Upon writing $\dfrac{1}{(2-z)^3}=\dfrac{-1}{(z-2)^3}$, we find that the principal part of $\dfrac{1}{(2-z)^3}$ at its isolated singular point $z=2$ is simply the function itself. That point is evidently a pole (of order 3).

This enables us to write

$$f(z) = \frac{1}{(z-ai)^3}\left[-a^2 i - \frac{a}{2}(z-ai) - \frac{i}{2}(z-ai)^2 + \cdots\right] \qquad (0 < |z - ai| < 2a).$$

The principal part of f at the point $z = ai$ is, then,

$$-\frac{i/2}{z-ai} - \frac{a/2}{(z-ai)^2} - \frac{a^2 i}{(z-ai)^3}.$$

SECTION 67

1. (*a*) The function $f(z) = \dfrac{z^2+2}{z-1}$ has an isolated singular point at $z = 1$. Writing $f(z) = \dfrac{\phi(z)}{z-1}$, where $\phi(z) = z^2 + 2$, and observing that $\phi(z)$ is analytic and nonzero at $z = 1$, we see that $z = 1$ is a pole of order $m = 1$ and that the residue there is $B = \phi(1) = 3$.

480

74

(*b*) If we write

$$f(z) = \left(\frac{z}{2z+1}\right)^3 = \frac{\phi(z)}{\left[z - \left(-\frac{1}{2}\right)\right]^3}, \quad \text{where} \quad \phi(z) = \frac{z^3}{8},$$

we see that $z = -\dfrac{1}{2}$ is a singular point of f. Since $\phi(z)$ is analytic and nonzero at that point, f has a pole of order $m = 3$ there. The residue is

$$B = \frac{\phi''(-1/2)}{2!} = -\frac{3}{16}.$$

(*c*) The function

$$\frac{\exp z}{z^2 + \pi^2} = \frac{\exp z}{(z - \pi i)(z + \pi i)}$$

has poles of order $m = 1$ at the two points $z = \pm \pi i$. The residue at $z = \pi i$ is

$$B_1 = \frac{\exp \pi i}{2\pi i} = \frac{-1}{2\pi i} = \frac{i}{2\pi},$$

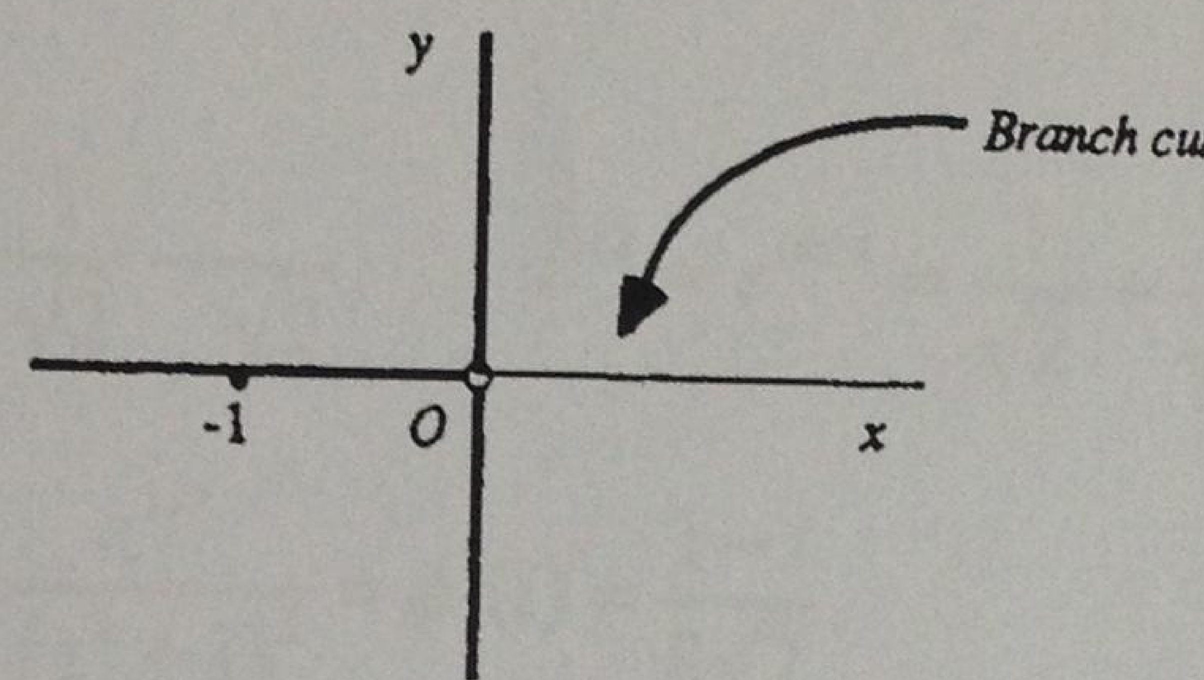
and the one at $z = -\pi i$ is

$$B_2 = \frac{\exp(-\pi i)}{-2\pi i} = \frac{-1}{-2\pi i} = -\frac{i}{2\pi}.$$

2. (a) Write the function $f(z)=\dfrac{z^{1/4}}{z+1}$ $(|z|>0, 0<\arg z<2\pi)$ as

$$f(z)=\frac{\phi(z)}{z+1}, \quad \text{where} \quad \phi(z)=z^{1/4}=e^{\frac{1}{4}\log z} \quad (|z|>0, 0<\arg z<2\pi).$$

The function $\phi(z)$ is analytic throughout its domain of definition, indicated in the figure below.



Also,

$$\phi(-1)=(-1)^{1/4}=e^{\frac{1}{4}\log(-1)}=e^{\frac{1}{4}(\ln 1+i\pi)}=e^{i\pi/4}=\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}=\frac{1+i}{\sqrt{2}}\neq 0.$$

This shows that the function f has a pole of order $m=1$ at $z=-1$, the residue there being

$$B=\phi(-1)=\frac{1+i}{\sqrt{2}}.$$

(b) Write the function $f(z)=\dfrac{\operatorname{Log} z}{(z^2+1)^2}$ as

$$f(z)=\frac{\phi(z)}{(z-i)^2} \quad \text{where} \quad \phi(z)=\frac{\operatorname{Log} z}{(z+i)^2}.$$

From this, it is clear that $f(z)$ has a pole of order $m=2$ at $z=i$. Straightforward differentiation then reveals that

$$\operatorname*{Res}_{z=i}\frac{\operatorname{Log} z}{(z^2+1)^2}=\phi'(i)=\frac{\pi+2i}{8}.$$

(c) Write the function

$$f(z) = \frac{z^{1/2}}{(z^2+1)^2} \qquad (|z| > 0, 0 < \arg z < 2\pi)$$

as

$$f(z) = \frac{\phi(z)}{(z-i)^2} \quad \text{where} \quad \phi(z) = \frac{z^{1/2}}{(z+i)^2}.$$

Since

$$\phi'(z) = \frac{(z+i)z^{-1/2} - 4z^{1/2}}{2(z+i)^3}$$

and

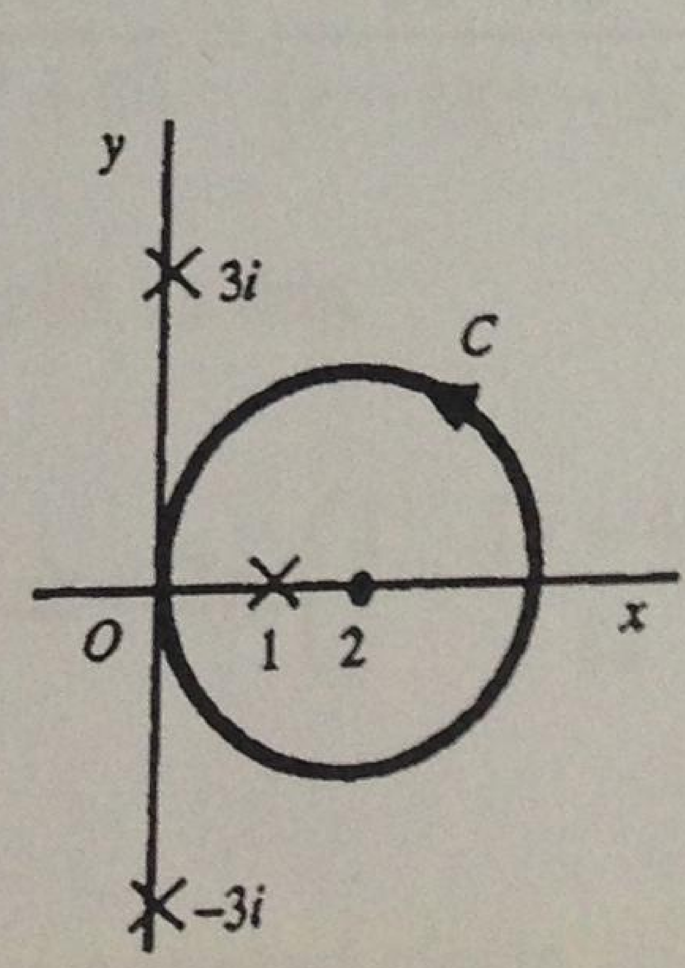
$$i^{-1/2} = e^{-i\pi/4} = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}, \qquad i^{1/2} = e^{i\pi/4} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}},$$

$$\operatorname*{Res}_{z=i} \frac{z^{1/2}}{(z^2+1)^2} = \phi'(i) = \frac{1-i}{8\sqrt{2}}.$$

3. (a) We wish to evaluate the integral

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)} dz,$$

where C is the circle $|z-2| = 2$, taken in the counterclockwise direction. That circle and the singularities $z = 1, \pm 3i$ of the integrand are shown in the figure just below.



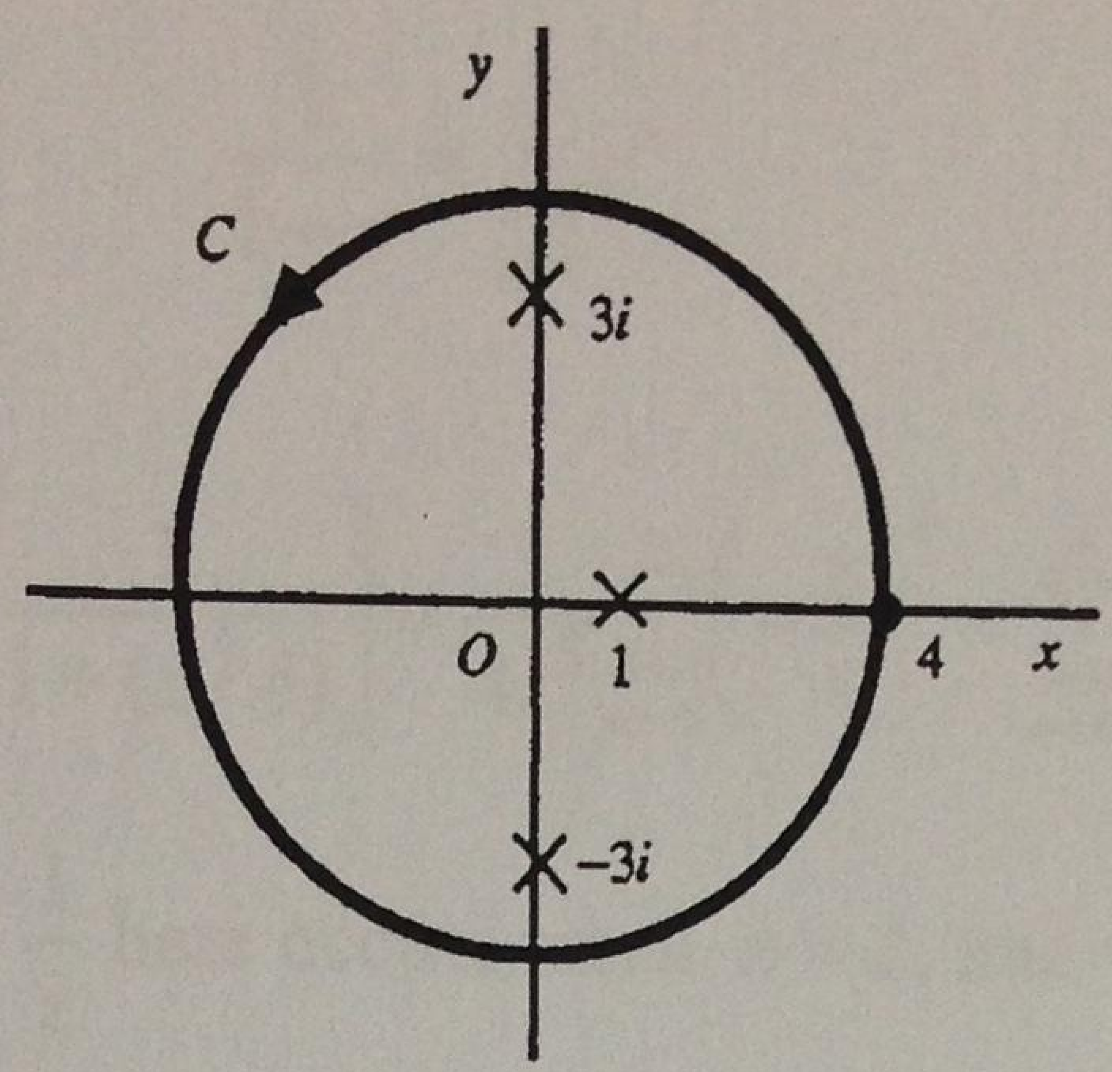
Observe that the point $z = 1$, which is the only singularity inside C, is a simple pole of the integrand and that

$$\operatorname*{Res}_{z=1} \frac{3z^3+2}{(z-1)(z^2+9)} = \left.\frac{3z^3+2}{z^2+9}\right]_{z=1} = \frac{1}{2}.$$

According to the residue theorem, then,

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)} dz = 2\pi i\left(\frac{1}{2}\right) = \pi i.$$

(b) Let us redo part *(a)* when C is changed to be the positively oriented circle $|z| = 4$, shown in the figure below.



In this case, all three singularities $z = 1, \pm 3i$ of the integrand are interior to C. We already know from part *(a)* that

$$\operatorname*{Res}_{z=1} \frac{3z^3 + 2}{(z-1)(z^2+9)} = \frac{1}{2}.$$

It is, moreover, straightforward to show that

$$\operatorname*{Res}_{z=3i} \frac{3z^3 + 2}{(z-1)(z^2+9)} = \left.\frac{3z^3 + 2}{(z-1)(z+3i)}\right]_{z=3i} = \frac{15+49i}{12}$$

and

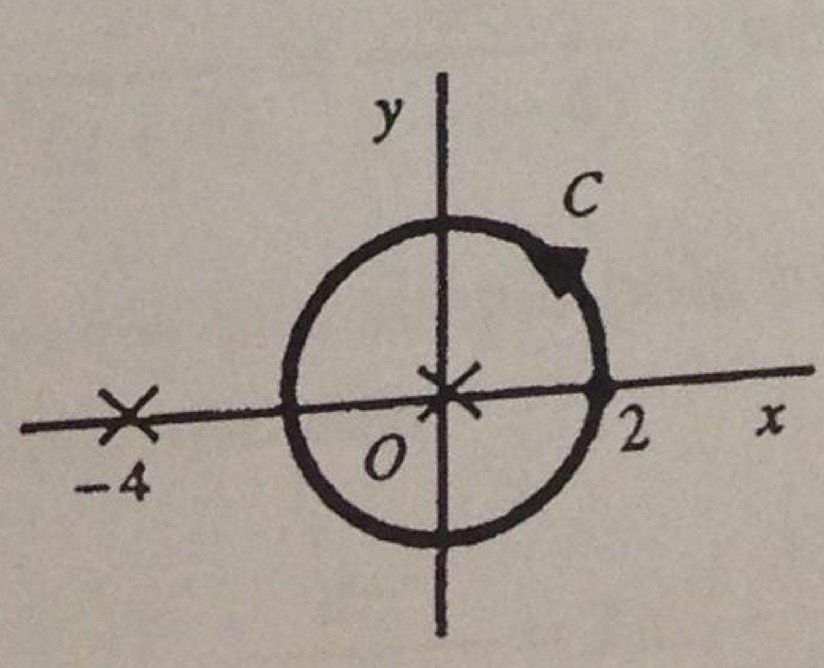
$$\operatorname*{Res}_{z=-3i} \frac{3z^3 + 2}{(z-1)(z^2+9)} = \left.\frac{3z^3 + 2}{(z-1)(z-3i)}\right]_{z=-3i} = \frac{15-49i}{12}.$$

The residue theorem now tells us that

$$\int_C \frac{3z^3 + 2}{(z-1)(z^2+9)}\,dz = 2\pi i\left(\frac{1}{2} + \frac{15+49i}{12} + \frac{15-49i}{12}\right) = 6\pi i.$$

4. *(a)* Let C denote the positively oriented circle $|z| = 2$, and note that the integrand of the integral $\int_C \frac{dz}{z^3(z+4)}$ has singularities at $z = 0$ and $z = -4$. (See the figure below.)

4
248
sec. 74



~~4 س~~
~~248.9~~
~~sec. 7A~~

To find the residue of the integrand at $z=0$, we recall the expansion

$$\frac{1}{1-z}=\sum_{n=0}^{\infty} z^n \qquad (|z|<1)$$

and write

$$\frac{1}{z^3(z+4)}=\frac{1}{4z^3}\left[\frac{1}{1+(z/4)}\right]=\frac{1}{4z^3}\sum_{n=0}^{\infty}\left(-\frac{z}{4}\right)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}z^{n-3} \qquad (0<|z|<4).$$

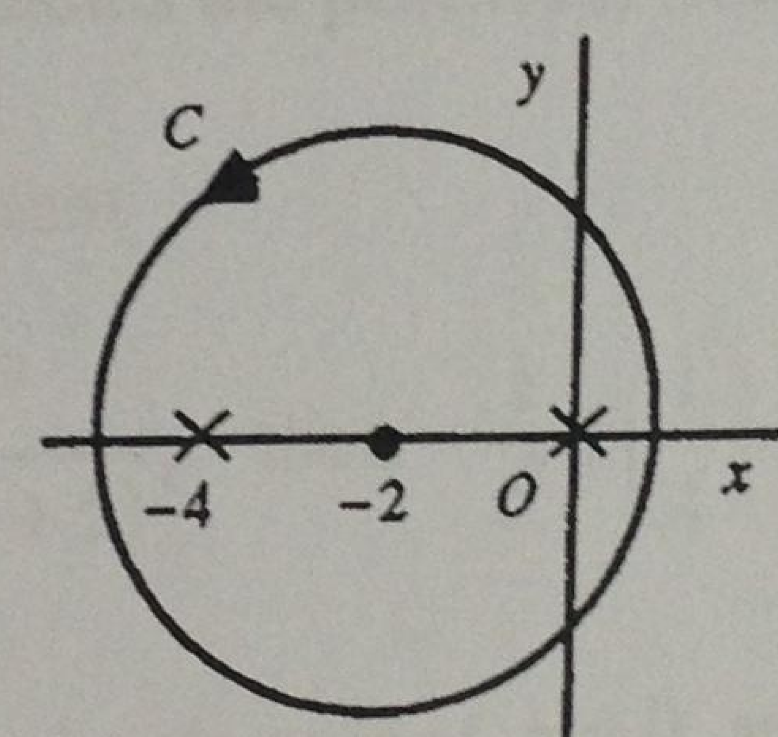
Now the coefficient of $\frac{1}{z}$ here occurs when $n=2$, and we see that

$$\operatorname*{Res}_{z=0}\frac{1}{z^3(z+4)}=\frac{1}{64}.$$

Consequently,

$$\int_C \frac{dz}{z^3(z+4)}=2\pi i\left(\frac{1}{64}\right)=\frac{\pi i}{32}.$$

(*b*) Let us replace the path C in part (*a*) by the positively oriented circle $|z+2|=3$, centered at -2 and with radius 3. It is shown below.



We already know from part (*a*) that

$$\operatorname*{Res}_{z=0}\frac{1}{z^3(z+4)}=\frac{1}{64}.$$

To find the residue at -4, we write

$$\frac{1}{z^3(z+4)}=\frac{\phi(z)}{z-(-4)}, \quad \text{where} \quad \phi(z)=\frac{1}{z^3}.$$

This tells us that $z=-4$ is a simple pole of the integrand and that the residue there is $\phi(-4)=-1/64$. Consequently,

$$\int_C \frac{dz}{z^3(z+4)}=2\pi i\left(\frac{1}{64}-\frac{1}{64}\right)=0.$$

5. Let us evaluate the integral $\int_C \frac{\cosh \pi z\, dz}{z(z^2+1)}$, where C is the positively oriented circle $|z|=2$.

5 80

248

sec.74

All three isolated singularities $z=0,\pm i$ of the integrand are interior to C. The desired residues are

$$\operatorname*{Res}_{z=0} \frac{\cosh \pi z}{z(z^2+1)} = \left.\frac{\cosh \pi z}{z^2+1}\right]_{z=0} = 1,$$

$$\operatorname*{Res}_{z=i} \frac{\cosh \pi z}{z(z^2+1)} = \left.\frac{\cosh \pi z}{z(z+i)}\right]_{z=i} = \frac{1}{2},$$

and

$$\operatorname*{Res}_{z=-i} \frac{\cosh \pi z}{z(z^2+1)} = \left.\frac{\cosh \pi z}{z(z-i)}\right]_{z=-i} = \frac{1}{2}.$$

Consequently,

$$\int_C \frac{\cosh \pi z\, dz}{z(z^2+1)} = 2\pi i\left(1+\frac{1}{2}+\frac{1}{2}\right) = 4\pi i.$$

SECTION 78

1. Write

$$\int_0^{2\pi}\frac{d\theta}{5+4\sin\theta}=\int_C \frac{1}{5+4\left(\dfrac{z-z^{-1}}{2i}\right)}\cdot\frac{dz}{iz}=\int_C\frac{dz}{2z^2+5iz-2},$$

where C is the positively oriented unit circle $|z|=1$. The quadratic formula tells us that the singular points of the integrand on the far right here are $z=-i/2$ and $z=-2i$. The point $z=-i/2$ is a simple pole interior to C; and the point $z=-2i$ is exterior to C. Thus

$$\int_0^{2\pi}\frac{d\theta}{5+4\sin\theta}=2\pi i\ \underset{z=-i/2}{\operatorname{Res}}\left[\frac{1}{2z^2+5iz-2}\right]=2\pi i\left[\frac{1}{4z+5i}\right]_{z=-i/2}=2\pi i\left(\frac{1}{3i}\right)=\frac{2\pi}{3}.$$

2. To evaluate the definite integral in question, write

$$\int_{-\pi}^{\pi}\frac{d\theta}{1+\sin^2\theta}=\int_C\frac{1}{1+\left(\dfrac{z-z^{-1}}{2i}\right)^2}\cdot\frac{dz}{iz}=\int_C\frac{4iz\,dz}{z^4-6z^2+1},$$

where C is the positively oriented unit circle $|z|=1$. This circle is shown below.