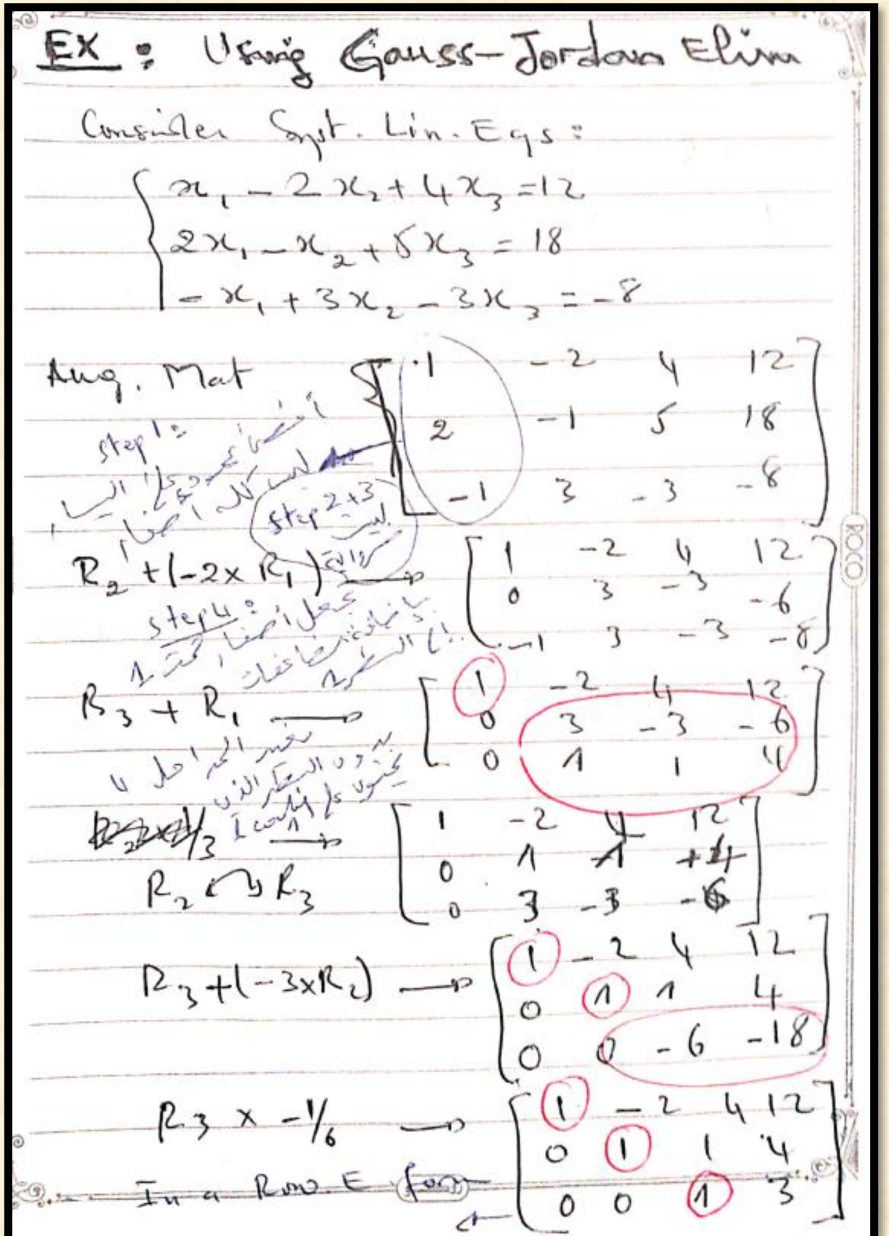
Further Application:



EX: Using Gauss-Jordan Elim

Consider Syst. Lin. Eqs:

$$\begin{cases} x_1 - 2x_2 + 4x_3 = 12 \\ 2x_1 - x_2 + 5x_3 = 18 \\ -x_1 + 3x_2 - 3x_3 = -8 \end{cases}$$

Aug. Mat $\left[\begin{array}{ccc|c} 1 & -2 & 4 & 12 \\ 2 & -1 & 5 & 18 \\ -1 & 3 & -3 & -8 \end{array}\right]$

step 1

step 2+3

$R_2 + (-2 \times R_1) \to \left[\begin{array}{cccc} 1 & -2 & 4 & 12 \\ 0 & 3 & -3 & -6 \\ -1 & 3 & -3 & -8 \end{array}\right]$

step 4

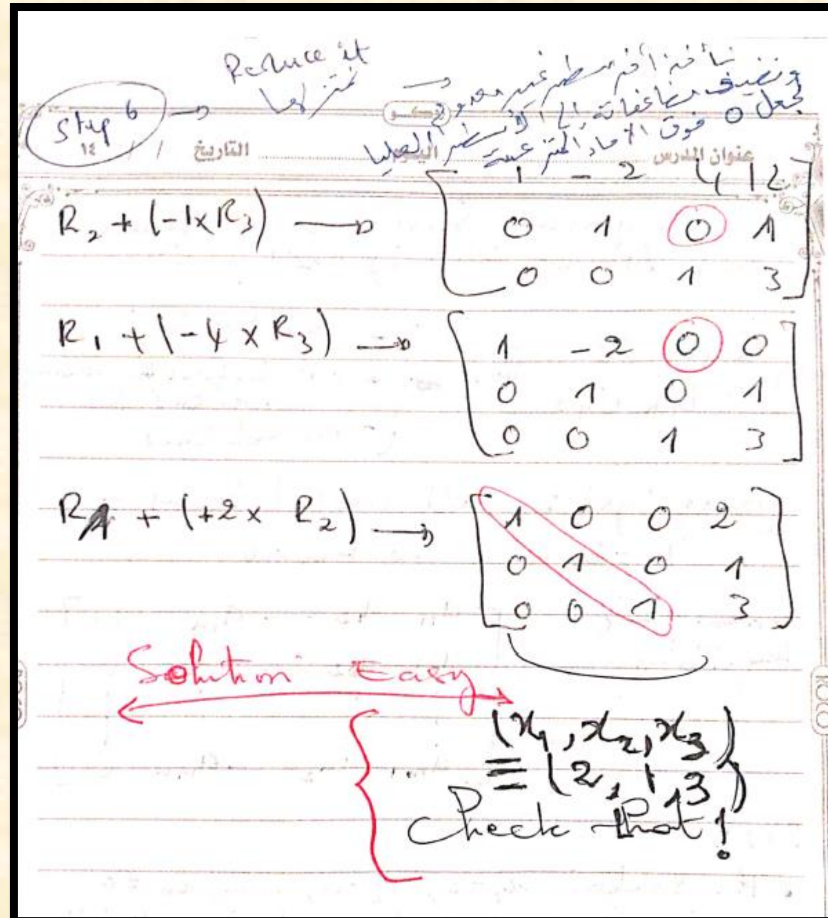
$R_3 + R_1 \to \left[\begin{array}{cccc} 1 & -2 & 4 & 12 \\ 0 & 3 & -3 & -6 \\ 0 & 1 & 1 & 4 \end{array}\right]$

$R_2 \leftrightarrow R_3 \to \left[\begin{array}{cccc} 1 & -2 & 4 & 12 \\ 0 & 1 & 1 & 4 \\ 0 & 3 & -3 & -6 \end{array}\right]$

$R_3 + (-3 \times R_2) \to \left[\begin{array}{cccc} 1 & -2 & 4 & 12 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & -6 & -18 \end{array}\right]$

$R_3 \times -\frac{1}{6} \to \left[\begin{array}{cccc} 1 & -2 & 4 & 12 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$

In a Row E form



Step 6 → Reduce it

$R_2 + (-1 \times R_3) \to \left[\begin{array}{cccc} 1 & -2 & 4 & 12 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$

$R_1 + (-4 \times R_3) \to \left[\begin{array}{cccc} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$

$R_1 + (+2 \times R_2) \to \left[\begin{array}{cccc} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$

Solution Easy

$(x_1, x_2, x_3) = (2, 1, 3)$

Check that!

Chapter 2

Matrices المصفوفات

2.1 Operations with Matrices

- Matrix: a rectangular array of numbers.

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \in M_{m\times n}$$

(i, j)-th entry: a_{ij}

row: m

column: n

size: $m \times n$

- *i*-th row vector

$$r_i = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix}$$

row matrix

- *j*-th column vector

$$c_j = \begin{bmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{mj} \end{bmatrix}$$

column matrix

- Square matrix: $m = n$

- **Diagonal matrix**: for square matrices

$$A = diag(d_1, d_2, \cdots, d_n) = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \in M_{n\times n}$$

- **Trace:**

If $A = [a_{ij}]_{n\times n}$

Then $Tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$

- Exp:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

$$\Rightarrow \quad r_1 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \quad r_2 = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}$$

$$\Rightarrow \quad c_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \; c_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \; c_3 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

- Equality of matrices:

If $A = [a_{ij}]_{m\times n}, \ B = [b_{ij}]_{m\times n}$

Then $A = B$ if and only if $a_{ij} = b_{ij} \ \forall 1 \le i \le m, \ 1 \le j \le n$

- Exp 1:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If $A = B$

Then $a = 1, \ b = 2, \ c = 3, \ d = 4$

- Matrix addition:

If $A=[a_{ij}]_{m\times n},\ B=[b_{ij}]_{m\times n}$ Condition: matrices of same size

Then $A+B=[a_{ij}]_{m\times n}+[b_{ij}]_{m\times n}=[a_{ij}+b_{ij}]_{m\times n}$ Sum of entries

ملاحظة: لايمكن جمع مصفوفات بأحجام مختلفة

- Exp 2: (Matrix addition)

$$\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}+\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}=\begin{bmatrix} -1+1 & 2+3 \\ 0-1 & 1+2 \end{bmatrix}=\begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}+\begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}=\begin{bmatrix} 1-1 \\ -3+3 \\ -2+2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Multiplication by a scalar:

  If $A=[a_{ij}]_{m\times n}$, c : scalar

  Then $cA=[ca_{ij}]_{m\times n}$

- Matrix subtraction:

  $A-B=A+(-1)B$

- Ex 3: (Scalar multiplication and matrix subtraction)

$$A=\begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \qquad B=\begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$

Find (*a*) 3*A*, (*b*) –*B*, (*c*) 3*A* – *B*

Sol:

(a)

$$3A = 3\begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3(1) & 3(2) & 3(4) \\ 3(-3) & 3(0) & 3(-1) \\ 3(2) & 3(1) & 3(2) \end{bmatrix} = \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix}$$

(b)

$$-B = (-1)\begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ -1 & 4 & -3 \\ 1 & -3 & -2 \end{bmatrix}$$

(c)

$$3A - B = \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 12 \\ -10 & 4 & -6 \\ 7 & 0 & 4 \end{bmatrix}$$

- Matrix multiplication:

$$\text{If } A=[a_{ij}]_{m\times n},\ B=[b_{ij}]_{n\times p}$$

$$\text{Then } AB=[a_{ij}]_{m\times n}[b_{ij}]_{n\times p}=[c_{ij}]_{m\times p}$$

Size of *matrix "AB"*

شرط الضرب:

عدد أعمدة الأولى يساوي عدد أسطر الثانية.

و الناتج حجمه "عدد أسطر الأولى ضرب عدد أعمدة الثانية"

$$\text{where } c_{ij}=\sum_{k=1}^{n}a_{ik}b_{kj}=a_{i1}b_{1j}+a_{i2}b_{2j}+\cdots+a_{in}b_{nj}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & \vdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} & & & & & \\ c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{in} \\ & & & & & \end{bmatrix}$$

- Note: (1) $A+B = B+A$, (2) $AB \neq BA$

- Ex 4: (Find *AB*)

$$A = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}$$

Sol:

$$AB = \begin{bmatrix} (-1)(-3)+(3)(-4) & (-1)(2)+(3)(1) \\ (4)(-3)+(-2)(-4) & (4)(2)+(-2)(1) \\ (5)(-3)+(0)(-4) & (5)(2)+(0)(1) \end{bmatrix}$$

$$= \begin{bmatrix} -9 & 1 \\ -4 & 6 \\ -15 & 10 \end{bmatrix}$$

- Matrix form of a system of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

m linear equations

$$\Downarrow$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\underset{\text{Coefficients matrix}}{A} \qquad \underset{\text{unknowns}}{x} \qquad b$$

Single matrix equation

$$\underset{m\times n}{A}\ \underset{n\times 1}{x} = \underset{m\times 1}{b}$$

- Partitioned matrices:

submatrix

$$A=\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right]=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$A=\left[\begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \\ \hline a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right]=\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

$$A=\left[\begin{array}{c|c|c|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right]=\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix}$$

Keywords in Section 2.1:

- row vector: متجه صفي
- column vector:متجه عمودي
- diagonal matrix: مصفوفة قطرية
- trace: اثر
- equality of matrices: تساوي أو تعادل المصفوفات
- matrix addition: جمع المصفوفات
- Multiplication by a scalar: ضرب بعدد ثابت
- matrix multiplication: ضرب المصفوفات
- partitioned matrix: مصفوفة مجزئة

2.2 Properties of Matrix Operations

- Three basic matrix operations:

  (1) matrix addition

  (2) scalar multiplication

  (3) matrix multiplication

- Zero matrix: $0_{m\times n}$

Of different orders

$$\begin{bmatrix}0&0\\0&0\end{bmatrix},\quad \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix},\quad \begin{bmatrix}0&0&0&0\\0&0&0&0\end{bmatrix},\quad \begin{bmatrix}0\\0\\0\\0\end{bmatrix},$$

- Identity matrix of order *n*: I_n

The identity matrix I_n is a n x n square matrix with the main diagonal of 1's and all other elements are 0's.

$$I_2=\begin{pmatrix}1&0\\0&1\end{pmatrix}\quad I_3=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\quad I_4=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

- Properties of matrix addition and scalar multiplication:

If $A, B, C \in M_{m\times n}$, c, d : scalar

Then (1) $A+B = B + A$

(2) $A + (B + C) = (A + B) + C$

(3) $(cd) A = c (dA)$

(4) $1A = A$

(5) $c(A+B) = cA + cB$

(6) $(c+d) A = cA + dA$

- Properties of zero matrices:

If $A \in M_{m\times n}$, $c : scalar$

Then (1) $A + 0_{m\times n} = A$

(2) $A + (-A) = 0_{m\times n}$

(3) $cA = 0_{m\times n} \Rightarrow c = 0 \; or \; A = 0_{m\times n}$

- Notes:

(1) $0_{m\times n}$: **the additive identity** for the set of all $m \times n$ matrices

(2) $-A$: **the additive inverse** of A

- **Transpose** of a matrix:

$$\text{If} \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in M_{m\times n}$$

$$\text{Then} \quad A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} \in M_{n\times m}$$

- Ex 8: (Find the **transpose** of the following matrix)

$$(a)\quad A=\begin{bmatrix}2\\8\end{bmatrix}\qquad (b)\quad A=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\qquad (c)\quad A=\begin{bmatrix}0&1\\2&4\\1&-1\end{bmatrix}$$

Sol: (a)

$$A=\begin{bmatrix}2\\8\end{bmatrix}\quad\Rightarrow A^T=\begin{bmatrix}2&8\end{bmatrix}$$

(b)

$$A=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\Rightarrow A^T=\begin{bmatrix}1&4&7\\2&5&8\\3&6&9\end{bmatrix}$$

(c)

$$A=\begin{bmatrix}0&1\\2&4\\1&-1\end{bmatrix}\quad\Rightarrow A^T=\begin{bmatrix}0&2&1\\1&4&-1\end{bmatrix}$$

- Properties of transposes:

(1) $(A^T)^T = A$

(2) $(A+B)^T = A^T + B^T$

(3) $(cA)^T = c(A^T)$

(4) $(AB)^T = B^T A^T$

- Symmetric matrix:

  A square matrix A is said **symmetric** if $A = A^T$

- Skew-symmetric matrix:

  A square matrix A is said **skew-symmetric** if $A^T = -A$

- Ex:

  If $A = \begin{bmatrix} 1 & 2 & 3 \\ a & 4 & 5 \\ b & c & 6 \end{bmatrix}$ is symmetric, find a, b, c?

Sol:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ a & 4 & 5 \\ b & c & 6 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & a & b \\ 2 & 4 & c \\ 3 & 5 & 6 \end{bmatrix}$$

$$A = A^T$$
$$\Rightarrow a = 2, b = 3, c = 5$$

- Ex:

If $A = \begin{bmatrix} 0 & 1 & 2 \\ a & 0 & 3 \\ b & c & 0 \end{bmatrix}$ is a skew-symmetric, find a, b, c?

Sol:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ a & 0 & 3 \\ b & c & 0 \end{bmatrix} \qquad -A^T = \begin{bmatrix} 0 & -a & -b \\ -1 & 0 & -c \\ -2 & -3 & 0 \end{bmatrix}$$

$$A = -A^T \ \Rightarrow \ a = -1, b = -2, c = -3$$

- Note: AA^T is symmetric

Pf: $(AA^T)^T = (A^T)^T A^T = AA^T$

$\therefore AA^T$ is symmetric

- Real number:

$ab = ba$ (Commutative law for multiplication)

- Matrix:

$\underset{m\times n}{A}\underset{n\times p}{B} \neq BA$ الترتيب مهم في ضرب المصفوفات

Three situations:

(1) If $m \neq p$, then AB is defined, BA is undefined.

(2) If $m = p, m \neq n$, then $AB \in M_{m\times m}, \; BA \in M_{n\times n}$ (Sizes are not the same)

(3) If $m = p = n$, then $AB \in M_{m\times m}, \; BA \in M_{m\times m}$

(Sizes are the same, but matrices are not equal)

- Ex 4:

check that AB and BA are not equal for the matrices.

$$A=\begin{bmatrix}1 & 3\\ 2 & -1\end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix}2 & -1\\ 0 & 2\end{bmatrix}$$

Sol:

$$AB=\begin{bmatrix}1 & 3\\ 2 & -1\end{bmatrix}\begin{bmatrix}2 & -1\\ 0 & 2\end{bmatrix}=\begin{bmatrix}2 & 5\\ 4 & -4\end{bmatrix}$$

$$BA=\begin{bmatrix}2 & -1\\ 0 & 2\end{bmatrix}\begin{bmatrix}1 & 3\\ 2 & -1\end{bmatrix}=\begin{bmatrix}0 & 7\\ 4 & -2\end{bmatrix}$$

- Note: $AB \neq BA$

- Real number:

$$ac = bc, \ c \neq 0$$

$$\Rightarrow \ a = b$$ (Cancellation law)

- Matrix:

$$AC = BC \quad C \neq 0$$

(1) If C is invertible, then $A = B$

(2) If C is not invertible, then $A \neq B$ *(Cancellation is not valid)*

- Ex 5: (An example in which cancellation is not valid)

Show that $AC=BC$

$$A=\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix},\quad B=\begin{bmatrix}2 & 4\\ 2 & 3\end{bmatrix},\quad C=\begin{bmatrix}1 & -2\\ -1 & 2\end{bmatrix}$$

Sol:

$$AC=\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & -2\\ -1 & 2\end{bmatrix}=\begin{bmatrix}-2 & 4\\ -1 & 2\end{bmatrix}$$

$$BC=\begin{bmatrix}2 & 4\\ 2 & 3\end{bmatrix}\begin{bmatrix}1 & -2\\ -1 & 2\end{bmatrix}=\begin{bmatrix}-2 & 4\\ -1 & 2\end{bmatrix}$$

So $AC=BC$

But $A\neq B$

Keywords in Section 2.2:

- zero matrix: مصفوفة صفرية
- identity matrix: مصفوفة الوحدة
- transpose matrix: مصفوفة منقولة
- symmetric matrix: مصفوفة متماثلة
- skew-symmetric matrix: مصفوفة متماثلة منحرفة

2.3 The Inverse of a Matrix

- Inverse matrix:

Consider $A \in M_{n\times n}$ (square matrix)

If there exists a matrix $B \in M_{n\times n}$ such that $AB = BA = I_n$,

Then (1) A is **invertible** (or **nonsingular**)

(2) B is **the inverse** of A

- Note:

A matrix that does not have an inverse is called **noninvertible** (or **singular**).

- Thm 2.7: (The inverse of a matrix is unique)

  If B and C are both inverses of the matrix A, **then $B = C$.**

Pf:

$$AB = I$$

$$C(AB) = CI$$

$$(CA)B = C$$

$$IB = C$$

$$B = C$$

  Consequently, the inverse of a matrix (if it exists) it is unique.

- Notes:

  (1) The inverse of A is denoted by A^{-1}

  (2) $AA^{-1} = A^{-1}A = I$

- Find the inverse of a matrix using Gauss-Jordan Elimination:

$$[A \mid I] \xrightarrow{\text{Gauss-Jordan Elimination}} [I \mid A^{-1}]$$

- Ex 2: (Find the inverse of the matrix)

$$A=\begin{bmatrix}1 & 4\\ -1 & -3\end{bmatrix}$$

Sol:

$$AX = I \qquad ; \quad \textbf{X?}$$

$$\begin{bmatrix}1 & 4\\ -1 & -3\end{bmatrix}\begin{bmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}x_{11}+4x_{21} & x_{12}+4x_{22}\\ -x_{11}-3x_{21} & -x_{12}-3x_{22}\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$\Rightarrow \begin{array}{rcrcl} x_{11} & + & 4x_{21} & = & 1 \\ -x_{11} & - & 3x_{21} & = & 0 \end{array} \qquad (1)$$

$$\begin{array}{rcrcl} x_{12} & + & 4x_{22} & = & 0 \\ -x_{12} & - & 3x_{22} & = & 1 \end{array} \qquad (2)$$

$$(1) \Rightarrow \left[\begin{array}{cc:c} 1 & 4 & 1 \\ -1 & -3 & 0 \end{array}\right] \xrightarrow{r_{12}^{(1)},\ r_{21}^{(-4)}} \left[\begin{array}{cc:c} 1 & 0 & -3 \\ 0 & 1 & 1 \end{array}\right] \Rightarrow x_{11} = -3, x_{21} = 1$$

$$(2) \Rightarrow \left[\begin{array}{cc:c} 1 & 4 & 0 \\ -1 & -3 & 1 \end{array}\right] \xrightarrow{r_{12}^{(1)},\ r_{21}^{(-4)}} \left[\begin{array}{cc:c} 1 & 0 & -4 \\ 0 & 1 & 1 \end{array}\right] \Rightarrow x_{12} = -4, x_{22} = 1$$

Thus

$$X = A^{-1} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} \quad (AX = I = AA^{-1})$$

- Note:

$$\begin{bmatrix} 1 & 4 & \vdots & 1 & 0 \\ -1 & -3 & \vdots & 0 & 1 \end{bmatrix} \xrightarrow[r_{12}^{(1)},\, r_{21}^{(-4)}]{\text{Gauss–Jordan Elimination}} \begin{bmatrix} 1 & 0 & \vdots & -3 & -4 \\ 0 & 1 & \vdots & 1 & 1 \end{bmatrix}$$

$$\quad A \qquad\qquad I \qquad\qquad\qquad\qquad I \qquad\qquad A^{-1}$$

If A can't be row reduced to I, then A is singular (noninvertible).

- Ex 3: (Find the inverse of the following matrix)

$$A=\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$$

خطوات حساب معكوس مصفوفة

1- يجب أن تكون المصفوفة قابلة للعكس

2- نقوم بمجموعة عمليات الصف الأولية لايجاد مصفوفة الوحدة

3- نقوم بنفس العمليات على مصفوفة الوحدة و بنفس الترتيب و الناتج هو معكوس المصفوفة

ملاحظة: اذا تم في أي خطوة من عمليات السطر البسيطة ايجاد سطر كله أصفار نتوقف و نقول المصفوفة غير قابلة للعكس

Sol:

$$[A \;\vdots\; I]=\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{r_{12}^{(-1)}}\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right]\xrightarrow{r_{13}^{(6)}}\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{r_{23}^{(4)}}\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{array}\right]\xrightarrow{r_{3}^{(-1)}}\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right]$$

$$\xrightarrow{r_{32}^{(1)}} \left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \xrightarrow{r_{21}^{(1)}} \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -1 & -4 & -1 \end{array}\right]$$

$$= [\, I \;\vdots\; A^{-1} \,]$$

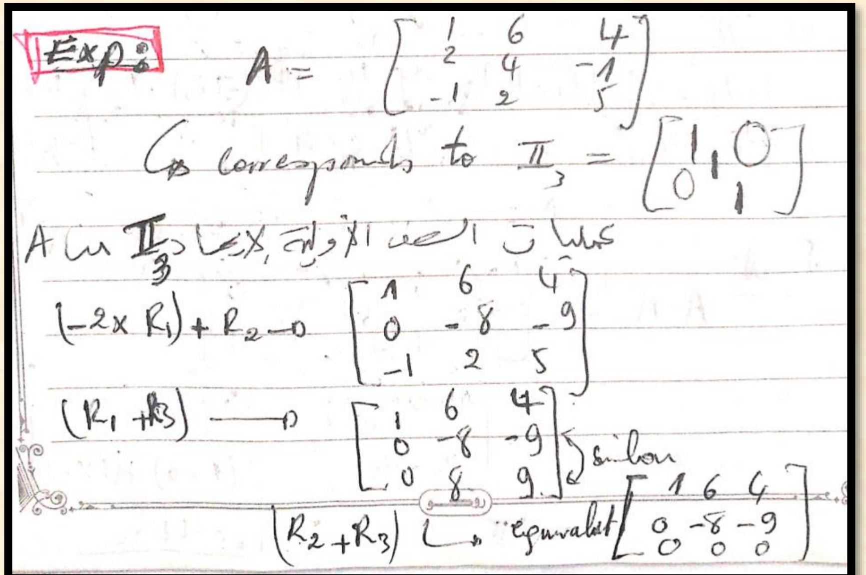
So the matrix A is invertible, and its inverse is

$$A^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}$$

- Check:

$$AA^{-1} = A^{-1}A = I$$

Further examples:



Exp: $A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$

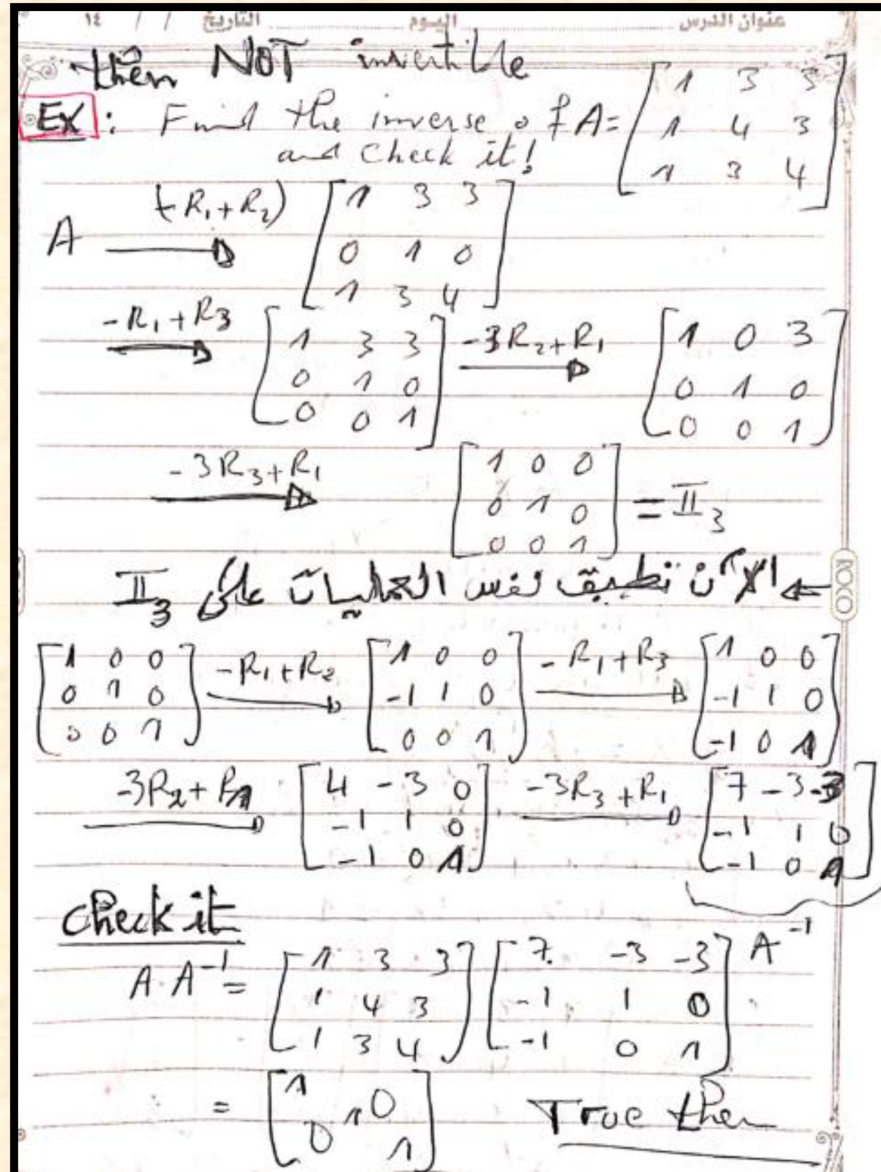
↳ corresponds to $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A \sim I_3$ بعمليات الصف الأولية ex,

$(-2 \times R_1) + R_2 \to \begin{bmatrix} 1 & 6 & 4 \\ 0 & -8 & -9 \\ -1 & 2 & 5 \end{bmatrix}$

$(R_1 + R_3) \to \begin{bmatrix} 1 & 6 & 4 \\ 0 & -8 & -9 \\ 0 & 8 & 9 \end{bmatrix}$ similar

$(R_2 + R_3)$ ↳ equivalent $\begin{bmatrix} 1 & 6 & 4 \\ 0 & -8 & -9 \\ 0 & 0 & 0 \end{bmatrix}$



then NOT invertible

Ex: Find the inverse of $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ and check it!

$A \xrightarrow{-R_1+R_2} \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 1 & 3 & 4 \end{bmatrix} \xrightarrow{-R_1+R_3} \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-3R_2+R_1} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\xrightarrow{-3R_3+R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$

← الآن نطبق نفس العمليات على I_3

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-R_1+R_2} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-R_1+R_3} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

$\xrightarrow{-3R_2+R_1} \begin{bmatrix} 4 & -3 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow{-3R_3+R_1} \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Check it

$A \cdot A^{-1} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ (A^{-1})

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ True then

- Power of a square matrix:

$$(1) A^0 = I$$

$$(2) A^k = \underbrace{AA\cdots A}_{k \text{ factors}} \quad (k > 0)$$

$$(3) A^r \cdot A^s = A^{r+s} \quad r, s : \text{integers}$$

$$(A^r)^s = A^{rs}$$

$$(4) D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \Rightarrow D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

للمصفوفات القطرية

- Thm 2.8： (Properties of the inverse matrices)

If A is an invertible matrix, k is a positive integer, and c is a scalar not equal to zero, then

(1) A^{-1} is invertible and $(A^{-1})^{-1} = A$

(2) A^{k} is invertible and $(A^{k})^{-1} = \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{k \text{ factors}} = (A^{-1})^{k} = A^{-k}$

(3) cA is invertible and $(cA)^{-1} = \frac{1}{c}A^{-1}, c \neq 0$

(4) A^{T} is invertible and $(A^{T})^{-1} = (A^{-1})^{T}$

- Thm 2.9: (The inverse of a product)

If A and B are invertible matrices of same size n, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Pf:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I)A^{-1} = (AI)A^{-1} = AA^{-1} = I$$

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}(I)B = B^{-1}(IB) = B^{-1}B = I$$

If AB is invertible, then its inverse is unique.

So $(AB)^{-1} = B^{-1}A^{-1}$

- **Note:**

$$(A_1A_2A_3\cdots A_n)^{-1} = A_n^{-1}\cdots A_3^{-1}A_2^{-1}A_1^{-1}$$

لاحظ تبديل الترتيب

- Thm 2.10 (Cancellation properties)

If C is an invertible matrix, then the following properties hold:

(1) If $AC=BC$, then $A=B$ (Right cancellation property)

(2) If $CA=CB$, then $A=B$ (Left cancellation property)

Pf:

$$AC = BC$$

$$(AC)C^{-1} = (BC)C^{-1} \quad \text{(C is invertible, so } C^{-1} \text{ exists)}$$

$$A(CC^{-1}) = B(CC^{-1})$$

$$AI = BI$$

$$A = B$$

- **Note:**

If C is not invertible, then cancellation is not valid.

- Thm 2.11: (Systems of equations with unique solutions)

If A is an invertible matrix, then the system of linear equations $Ax = b$ has a unique solution given by $x = A^{-1}b$

Pf:
$$Ax = b$$
$$A^{-1}Ax = A^{-1}b \quad (A \text{ is nonsingular})$$
$$Ix = A^{-1}b$$
$$x = A^{-1}b$$

If x_1 and x_2 were two solutions of equation $Ax = b$.

then $Ax_1 = b = Ax_2 \Rightarrow x_1 = x_2$ (Left cancellation property)

This solution is unique.

- Note:

For square systems (those having the same number of equations as variables), Theorem 2.11 can be used to determine whether the system has a unique solution.

- Note:

$Ax = b$ (A is an invertible matrix)

$$\left[A \mid b\right] \xrightarrow{A^{-1}} \left[A^{-1}A \mid A^{-1}b\right] = \left[I \mid A^{-1}b\right]$$

Keywords in Section 2.3:

- inverse matrix: مصفوفة عكسية
- invertible: قابلة للعكس
- nonsingular: غير منفرد
- singular: منفرد
- power: قوة أو أس

2.4 Elementary Matrices

- Row elementary matrix:

  An ***n×n*** matrix is called an *Elementary Matrix* if it can be obtained from the identity matrix $\boldsymbol{I_n}$ by a single elementary operation.

- Three row elementary matrices:

  (1) $R_{ij} = r_{ij}(I)$ Interchange two rows.

  (2) $R_i^{(k)} = r_i^{(k)}(I)$ $(k \neq 0)$ Multiply a row by a nonzero constant.

  (3) $R_{ij}^{(k)} = r_{ij}^{(k)}(I)$ Add a multiple of a row to another row.

- Note:

  Only do **a single** elementary row operation.

- Ex 1: (Elementary matrices and nonelementary matrices)

$$(a)\begin{bmatrix}1&0&0\\0&3&0\\0&0&1\end{bmatrix}$$

Yes $(r_2^{(3)}(I_3))$

$$(b)\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}$$

No (not square)

$$(c)\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$$

No (Row multiplication must be by a nonzero constant)

$$(d)\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$$

Yes $(r_{23}(I_3))$

$$(e)\begin{bmatrix}1&0\\2&1\end{bmatrix}$$

Yes $(r_{12}^{(2)}(I_2))$

$$(f)\begin{bmatrix}1&0&0\\0&2&0\\0&0&-1\end{bmatrix}$$

No (Use two elementary row operations)

- Remember:

  Only do **a single** elementary row operation.

- Thm 2.12: (Representing elementary row operations)

Let ***E*** be the elementary matrix obtained by performing an elementary row operation on $\boldsymbol{I_m}$. If that same elementary row operation is performed on an ***m×n*** matrix ***A***, then the resulting matrix is given by the product ***EA***. (*E is mxm square matrix*)

$$r(I) = E$$
$$r(A) = EA$$

- Notes:

(1) $r_{ij}(A) = R_{ij}A$

(2) $r_i^{(k)}(A) = R_i^{(k)}A$

(3) $r_{ij}^{(k)}(A) = R_{ij}^{(k)}A$

ناتج تطبيق نفس العملية على المصفوفة الثانية

- Ex 2: (Elementary matrices and elementary row operation)

$$(a)\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&2&1\\1&-3&6\\3&2&-1\end{bmatrix}=\begin{bmatrix}1&-3&6\\0&2&1\\3&2&-1\end{bmatrix}(r_{12}(A)=R_{12}A)$$

$$(b)\begin{bmatrix}1&0&0\\0&\frac{1}{2}&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&-4&1\\0&2&6&-4\\0&1&3&1\end{bmatrix}=\begin{bmatrix}1&0&-4&1\\0&1&3&-2\\0&1&3&1\end{bmatrix}(r_2^{(\frac{1}{2})}(A)=R_2^{(\frac{1}{2})}A)$$

$$(c)\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&-1\\-2&-2&3\\0&4&5\end{bmatrix}=\begin{bmatrix}1&0&-1\\0&-2&1\\0&4&5\end{bmatrix}(r_{12}^{(2)}(A)=R_{12}^{(5)}A)$$

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- **Ex 3: (**Using elementary matrices)

Find a sequence of elementary matrices that can be used to write the matrix A in row-echelon form.

$$A = \begin{bmatrix} 0 & 1 & 3 & 5 \\ 1 & -3 & 0 & 2 \\ 2 & -6 & 2 & 0 \end{bmatrix}$$

Matrix *3*x*4*; we use then I_3

And apply Elem Row Oper that transform A (to R.E.F) to I_3

Sol:

$$E_1 = r_{12}(I_3) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2 = r_{13}^{(-2)}(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$E_3 = r_3^{(\frac{1}{2})}(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$A_1 = r_{12}(A) = E_1 A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 3 & 5 \\ 1 & -3 & 0 & 2 \\ 2 & -6 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 2 & -6 & 2 & 0 \end{bmatrix}$$

$$A_2 = r_{13}^{(-2)}(A_1) = E_2 A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 2 & -6 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & -4 \end{bmatrix}$$

$$A_3 = r_3^{(\frac{1}{2})}(A_2) = E_3 A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & -4 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix} = B$$

row-echelon form

$$\therefore \boxed{B = E_3 E_2 E_1 A} \quad \text{or} \quad B = r_3^{(\frac{1}{2})}(r_{13}^{(-2)}(r_{12}(A)))$$

- Row-equivalent:

Matrix ***B*** is **row-equivalent** to ***A*** if there exists a finite number of elementary matrices such that

$$B = E_k E_{k-1} \cdots E_2 E_1 A$$

- Thm 2.13: (Elementary matrices are invertible)

  If E is an elementary matrix, then E^{-1} exists and is an elementary matrix.

  - Important Note:

$$(1)\ \ (R_{ij})^{-1} = R_{ij}$$

$$(2)\ \ (R_i^{(k)})^{-1} = R_i^{(\frac{1}{k})}$$

$$(3)\ \ (R_{ij}^{(k)})^{-1} = R_{ij}^{(-k)}$$

- Ex:

Elementary Matrix — Inverse Matrix

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_{12} \qquad (R_{12})^{-1} = E_1^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_{12}$$ (Elementary Matrix)

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = R_{13}^{(-2)} \qquad (R_{13}^{(-2)})^{-1} = E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = R_{13}^{(2)}$$ (Elementary Matrix)

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = R_3^{(\frac{1}{2})} \qquad (R_3^{(\frac{1}{2})})^{-1} = E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = R_3^{(2)}$$ (Elementary Matrix)

(1) $(R_{ij})^{-1} = R_{ij}$

(2) $(R_i^{(k)})^{-1} = R_i^{(\frac{1}{k})}$

(3) $(R_{ij}^{(k)})^{-1} = R_{ij}^{(-k)}$

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الأحادية باحترام ما سبق ذكره

- Thm 2.14: (A property of invertible matrices)

A square matrix A is invertible **if and only if** it can be written as the product of elementary matrices.

Pf: (1) Assume that A is the product of elementary matrices.

(a) Every elementary matrix is invertible.

(b) The product of invertible matrices is invertible.

Thus A is invertible.

(2) If A is invertible, $A\mathbf{x} = 0$ has only the trivial solution. (Thm. 2.11)

$$\Rightarrow [A \vdots 0] \rightarrow [I \vdots 0]$$

$$\Rightarrow E_k \cdots E_3 E_2 E_1 A = I$$

$$\Rightarrow \underline{A = E_1^{-1} E_2^{-1} E_3^{-1} \cdots E_k^{-1}}$$

Thus A can be written as the product of elementary matrices.

- Ex 4:

Find a sequence of elementary matrices whose product is

$$A=\begin{bmatrix}-1 & -2\\ 3 & 8\end{bmatrix}$$

Sol:

$$A=\begin{bmatrix}-1 & -2\\ 3 & 8\end{bmatrix}\xrightarrow{r_1^{(-1)}}\begin{bmatrix}1 & 2\\ 3 & 8\end{bmatrix}\xrightarrow{r_{12}^{(-3)}}\begin{bmatrix}1 & 2\\ 0 & 2\end{bmatrix}$$

$$\xrightarrow{r_2^{(\frac{1}{2})}}\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\xrightarrow{r_{21}^{(-2)}}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I$$

Therefore $R_{21}^{(-2)}R_2^{(\frac{1}{2})}R_{12}^{(-3)}R_1^{(-1)}A=I$ *Note the order*

Thus $A=(R_1^{(-1)})^{-1}(R_{12}^{(-3)})^{-1}(R_2^{(\frac{1}{2})})^{-1}(R_{21}^{(-2)})^{-1}$

$$=R_1^{(-1)}R_{12}^{(3)}R_2^{(2)}R_{21}^{(2)}$$

$$=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 3 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$$

Remember

(1) $(R_{ij})^{-1}=R_{ij}$

(2) $(R_i^{(k)})^{-1}=R_i^{(\frac{1}{k})}$

(3) $(R_{ij}^{(k)})^{-1}=R_{ij}^{(-k)}$

- Important Note:

If A is invertible

Then $E_k\cdots E_3E_2E_1A=I$

$$A^{-1}=E_k\cdots E_3E_2E_1$$

$$E_k\cdots E_3E_2E_1[A\,\vdots\,I]=[I\,\vdots\,A^{-1}]$$

- Thm 2.15: (Equivalent conditions)

If A is an $\boldsymbol{n \times n}$ matrix, then the following statements are equivalent.

(1) A is invertible.

(2) $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n\times 1$ column matrix $\mathbf{b}$.

(3) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(4) A is row-equivalent to I_n.

(5) A can be written as the product of elementary matrices.