Chapter 7
Eigenvalues and Eigenvectors

7.1 Eigenvalues and Eigenvectors

- Eigenvalue problem:

  If A is an $n \times n$ matrix, do there exist nonzero vectors $\boldsymbol{x}$ in R^n such that $A\boldsymbol{x}$ is a scalar multiple of $\boldsymbol{x}$？

- Eigenvalue and eigenvector:

  A：an $n \times n$ matrix

  λ：a scalar

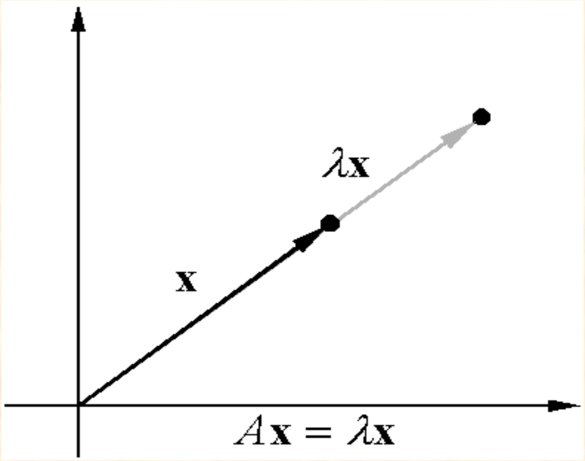
  $\boldsymbol{x}$： a nonzero vector in R^n

Eigenvalue

$$Ax = \lambda x$$

Eigenvector

- Geometrical Interpretation



- Ex 1: (Verifying eigenvalues and eigenvectors)

$$A=\begin{bmatrix}2 & 0\\0 & -1\end{bmatrix}\quad x_1=\begin{bmatrix}1\\0\end{bmatrix}\quad x_2=\begin{bmatrix}0\\1\end{bmatrix}$$

$$Ax_1=\begin{bmatrix}2 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix}=2\begin{bmatrix}1\\0\end{bmatrix}=2x_1$$

Eigenvalue: 2 ; Eigenvector: $\begin{bmatrix}1\\0\end{bmatrix}$

$$Ax_2=\begin{bmatrix}2 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\-1\end{bmatrix}=-1\begin{bmatrix}0\\1\end{bmatrix}=(-1)x_2$$

Eigenvalue: -1 ; Eigenvector: $\begin{bmatrix}0\\1\end{bmatrix}$

- Thm 7.1: (The eigenspace of A corresponding to λ)

  If A is an $n\times n$ matrix with an eigenvalue λ, then the set of all eigenvectors of λ together with the zero vector is a subspace of R^n. This subspace is called **the eigenspace of λ** .

Pf:

x_1 and x_2 are eigenvectors corresponding to λ

$(i.e.\ \ Ax_1=\lambda x_1,\ \ Ax_2=\lambda x_2)$

(1) $A(x_1+x_2)=Ax_1+Ax_2=\lambda x_1+\lambda x_2=\lambda(x_1+x_2)$

$(i.e.\ \ x_1+x_2$ is an eigenvector corresponding to $\lambda)$

(2) $A(cx_1)=c(Ax_1)=c(\lambda x_1)=\lambda(cx_1)$

$(i.e.\ \ cx_1$ is an eigenvector corresponding to $\lambda)$

- Ex 3: (An example of eigenspaces in the plane)

Find the eigenvalues and corresponding eigenspaces of

$$A=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$$

Sol:

If $\quad \text{v}=(x, y)$

$$A\text{v}=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}-x\\ y\end{bmatrix}$$

For a vector on the *x*-axis

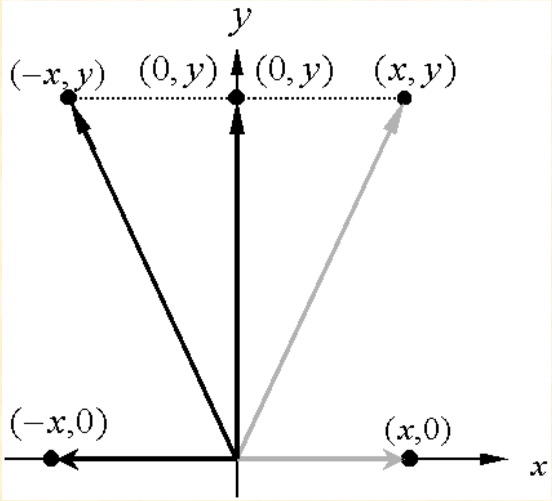
Eigenvalue $\lambda_1=-1$

$$\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x\\ 0\end{bmatrix}=\begin{bmatrix}-x\\ 0\end{bmatrix}=-1\begin{bmatrix}x\\ 0\end{bmatrix}$$

For a vector on the *y*-axis

Eigenvalue $\lambda_2 = 1$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix} = 1 \begin{bmatrix} 0 \\ y \end{bmatrix}$$



Geometrically, multiplying a vector (x, y) in R^2 by the matrix *A* corresponds to a reflection in the *y*-axis.

The eigenspace corresponding to $\lambda_1 = -1$ is the *x*-axis.

The eigenspace corresponding to $\lambda_2 = 1$ is the *y*-axis.

- Thm 7.2: (Finding eigenvalues and eigenvectors of a matrix $A \in M_{n\times n}$)

  Let A is an $n\times n$ matrix.

  (1) An eigenvalue of A is a scalar λ such that $\det(\lambda I - A) = 0$.

  (2) The eigenvectors of A corresponding to λ are the nonzero solutions of $(\lambda I - A)x = 0$.

- Note:

  $Ax = \lambda x \;\Rightarrow\; (\lambda I - A)x = 0$ (homogeneous system)

  If $(\lambda I - A)x = 0$ has nonzero solutions iff $\det(\lambda I - A) = 0$.

- Characteristic polynomial of $A \in M_{n\times n}$:

$$\det(\lambda I - A) = |(\lambda I - A)| = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

- Characteristic equation of A:

$$\det(\lambda I - A) = 0$$

- Ex 4: (Finding eigenvalues and eigenvectors)

$$A=\begin{bmatrix}2 & -12\\ 1 & -5\end{bmatrix}$$

Sol: Characteristic equation:

$$\det(\lambda I - A)=\begin{vmatrix}\lambda-2 & 12\\ -1 & \lambda+5\end{vmatrix}$$

$$=\lambda^2+3\lambda+2=(\lambda+1)(\lambda+2)=0$$

$$\Rightarrow \lambda=-1,-2$$

$$\text{Eigenvalues}: \ \lambda_1=-1, \lambda_2=-2$$

The eigenspace of A :

$$(1)\lambda_1 = -1 \Rightarrow (\lambda_1 I - A)x = \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\because \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix} = t \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \ t \neq 0$$

$$(2)\lambda_2 = -2 \Rightarrow (\lambda_2 I - A)x = \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\because \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \ t \neq 0$$

$$\text{Check}: \ Ax = \lambda_i x$$

- Ex 5: (Finding eigenvalues and eigenvectors)

Find the eigenvalues and corresponding eigenvectors for the matrix A. What is the dimension of the eigenspace of each eigenvalue?

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda \mathrm{I} - A| = \begin{vmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^3 = 0$$

Eigenvalue: $\lambda = 2$

The eigenspace of A corresponding to $\lambda = 2$:

$$(\lambda I - A)x = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\because \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ s, t \neq 0$$

$$\left\{ s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \middle| s, t \in R \right\} : \text{the eigenspace of A corresponding to } \lambda = 2$$

Thus, the dimension of its eigenspace is 2.

- Notes:

  (1) If an eigenvalue λ_1 occurs as a multiple root (*k times*) for the characteristic polynominal, then λ_1 has multiplicity *k*.

  (2) The multiplicity of an eigenvalue is greater than or equal to the dimension of its eigenspace.

- Ex 6：Find the eigenvalues of the matrix A and find a basis for each of the corresponding eigenspaces.

$$A=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 5 & -10\\ 1 & 0 & 2 & 0\\ 1 & 0 & 0 & 3\end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda \mathrm{I}-A|=\begin{vmatrix}\lambda-1 & 0 & 0 & 0\\ 0 & \lambda-1 & -5 & 10\\ -1 & 0 & \lambda-2 & 0\\ -1 & 0 & 0 & \lambda-3\end{vmatrix}$$

$$=(\lambda-1)^2(\lambda-2)(\lambda-3)=0$$

$$\text{Eigenvalues}: \lambda_1=1, \lambda_2=2, \lambda_3=3$$

$$(1)\lambda_1 = 1$$

$$\Rightarrow (\lambda_1 I - A)x = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 10 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 10 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2t \\ s \\ 2t \\ t \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \ \ s, t \neq 0$$

$$\Rightarrow \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$$ is a basis for the eigenspace of A corresponding to $\lambda = 1$

$$(2)\lambda_2 = 2 \quad \Rightarrow (\lambda_2 \mathrm{I} - A)x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 10 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 10 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \; t \neq 0$$

$$\Rightarrow \left\{ \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix} \right\}$$ is a basis for the eigenspace of A corresponding to $\lambda = 2$

$$(3)\lambda_3 = 3$$

$$\Rightarrow (\lambda_3 \mathrm{I} - A)x = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & -5 & 10 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & -5 & 10 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -5t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}, \ t \neq 0$$

$$\Rightarrow \left\{ \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$$ is a basis for the eigenspace of A corresponding to $\lambda = 3$

- Thm 7.3: (Eigenvalues of triangular matrices)

  If A is an $n\times n$ triangular matrix, then its eigenvalues are the entries on its main diagonal.

- Ex 7: (Finding eigenvalues for diagonal and triangular matrices)

$$(a) A=\begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 3 & -3 \end{bmatrix} \quad (b) A=\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Sol:

$$(a)\ |\lambda \mathrm{I} - A| = \begin{vmatrix} \lambda-2 & 0 & 0 \\ 1 & \lambda-1 & 0 \\ -5 & -3 & \lambda+3 \end{vmatrix} = (\lambda-2)(\lambda-1)(\lambda+3)$$

$$\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = -3$$

$$(b)\ \lambda_1 = -1, \lambda_2 = 2, \lambda_3 = 0, \lambda_4 = -4, \lambda_5 = 3$$

- Eigenvalues and eigenvectors of linear transformations:

A number λ is called an eigenvalue of a linear transformation $T: V \rightarrow V$ if there is a nonzero vector $\mathbf{x}$ such that $T(\mathbf{x}) = \lambda\mathbf{x}$. The vector $\mathbf{x}$ is called an eigenvector of T corresponding to λ, and the setof all eigenvectors of λ (with the zero vector) is called the eigenspace of λ.

- Ex 8: (Finding eigenvalues and eigenspaces)

Find the eigenvalues and corresponding eigenspaces

$$A=\begin{bmatrix}1 & 3 & 0\\ 3 & 1 & 0\\ 0 & 0 & -2\end{bmatrix}.$$

Sol:

$$|\lambda I - A| = \begin{vmatrix}\lambda-1 & -3 & 0\\ -3 & \lambda-1 & 0\\ 0 & 0 & \lambda+2\end{vmatrix} = (\lambda+2)^2(\lambda-4)$$

$$\text{eigenvalues}: \lambda_1 = 4,\ \lambda_2 = -2$$

The eigenspaces for these two eigenvalues are as follows.

$B_1 = \{(1, 1, 0)\}$ Basis for $\lambda_1 = 4$

$B_2 = \{(1, -1, 0), (0, 0, 1)\}$ Basis for $\lambda_2 = -2$

- Notes:

(1) Let $T{:}R^3 \rightarrow R^3$ be the linear transformation whose standard matrix is A in Ex. 8, and let B' be the basis of R^3 made up of three linear independent eigenvectors found in Ex. 8. Then A', the matrix of T relative to the basis B', is diagonal.

$$B' = \{(1, 1, 0), (1, -1, 0), (0, 0, 1)\}$$

Eigenvectors of A

$$A' = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Eigenvalues of A

(2) The main diagonal entries of the matrix A' are the eigenvalues of A.

7.2 Diagonalization

- Diagonalization problem:

  For a square matrix A, does there exist an invertible matrix P such that $P^{-1}AP$ is diagonal?

- Diagonalizable matrix:

  A square matrix A is called **diagonalizable** if there exists an invertible matrix P such that $P^{-1}AP$ is **a diagonal matrix**.

  (P diagonalizes A)

- Notes:

  (1) If there exists an invertible matrix P such that $B = P^{-1}AP$, then two square matrices A and B are called **similar**.

  (2) The eigenvalue problem is related closely to the diagonalization problem.

- Thm 7.4: (Similar matrices have the same eigenvalues)

  If A and B are similar $n \times n$ matrices, then they have the same eigenvalues.

Pf:

$$A \text{ and } B \text{ are similar} \Rightarrow B = P^{-1}AP$$

$$\begin{aligned}
|\lambda \mathrm{I} - B| &= |\lambda \mathrm{I} - P^{-1}AP| = |P^{-1}\lambda \mathrm{I}P - P^{-1}AP| = |P^{-1}(\lambda \mathrm{I} - A)P| \\
&= |P^{-1}||\lambda \mathrm{I} - A||P| = |P^{-1}||P||\lambda \mathrm{I} - A| = |P^{-1}P||\lambda \mathrm{I} - A| \\
&= |\lambda \mathrm{I} - A|
\end{aligned}$$

A and B have the same characteristic polynomial.

Thus A and B have the same eigenvalues.

- Ex 1: (A diagonalizable matrix)

$$A=\begin{bmatrix}1 & 3 & 0\\ 3 & 1 & 0\\ 0 & 0 & -2\end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda \mathrm{I}-A|=\begin{vmatrix}\lambda-1 & -3 & 0\\ -3 & \lambda-1 & 0\\ 0 & 0 & \lambda+2\end{vmatrix}=(\lambda-4)(\lambda+2)^2=0$$

$$\text{Eigenvalues}:\ \lambda_1=4,\ \lambda_2=-2,\ \lambda_3=-2$$

$$(1)\lambda=4 \Rightarrow \text{Eigenvector}:\ p_1=\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}\ \text{(See p.403 Ex.5)}$$

$$(2)\lambda = -2 \Rightarrow \text{Eigenvector}: p_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},\ p_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ (See p.403 Ex.5)}$$

$$P = [p_1 \quad p_2 \quad p_3] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- Notes:

$$(1)\ P = [p_2 \quad p_1 \quad p_3] = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$(2)\ P = [p_2 \quad p_3 \quad p_1] = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

- Thm 7.5: (Condition for diagonalization)

  An $n \times n$ matrix A is diagonalizable if and only if

  it has n linearly independent eigenvectors.

Pf:

$(\Rightarrow) A$ is diagonalizable

there exists an invertible P s.t. $D = P^{-1}AP$ is diagonal

Let $P = [\, p_1 \mid p_2 \mid \cdots \mid p_n \,]$ and $D = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$

$$PD = [\, p_1 \mid p_2 \mid \cdots \mid p_n \,]\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$= [\, \lambda_1 p_1 \mid \lambda_2 p_2 \mid \cdots \mid \lambda_n p_n \,]$$

$$AP = A[\, p_1 \mid p_2 \mid \cdots \mid p_n \,] = [\, Ap_1 \mid Ap_2 \mid \cdots \mid Ap_n \,]$$

$\because AP = PD$

$\therefore Ap_i = \lambda_i p_i, \; i = 1, 2, \ldots, n$

($i.e.$ the column vector p_i of P are eigenvectors of A)

$\because P$ is invertible $\Rightarrow p_1, p_2, \cdots, p_n$ are linearly independent.

$\therefore$ A has n linearly independent eigenvectors.

$(\Leftarrow)$ A has n linearly independent eigenvectors $p_1, p_2, \cdots p_n$

with corresponding eigenvalues $\lambda_1, \lambda_2, \cdots \lambda_n$

i.e. $Ap_i = \lambda_i p_i, \; i = 1, 2, \ldots, n$

Let $P = [\, p_1 \mid p_2 \mid \cdots \mid p_n \,]$

$$
\begin{aligned}
AP &= A[\,p_1 \mid p_2 \mid \cdots \mid p_n\,] \\
&= [\,Ap_1 \mid Ap_2 \mid \cdots \mid Ap_n\,] \\
&= [\,\lambda_1 p_1 \mid \lambda_2 p_2 \mid \cdots \mid \lambda_n p_n\,] \\
&= [\,p_1 \mid p_2 \mid \cdots \mid p_n\,]\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = PD
\end{aligned}
$$

$\because\ p_1, p_1, \cdots, p_n$ are linearly independent $\Rightarrow P$ is invertible

$\therefore\ P^{-1}AP = D$

$\Rightarrow A$ is diagonalizable

Note: If n linearly independent vectors do not exist, then an $n\times n$ matrix A is not diagonalizable.

- Ex 4: (A matrix that is not diagonalizable)

Show that the following matrix is not diagonalizable.

$$A=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix}\lambda-1 & -2\\ 0 & \lambda-1\end{vmatrix} = (\lambda-1)^2 = 0$$

$$\text{Eigenvalue}: \ \lambda_1 = 1$$

$$\lambda I - A = I - A = \begin{bmatrix}0 & -2\\ 0 & 0\end{bmatrix} \sim \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} \Rightarrow \text{Eigenvector}: \ p_1 = \begin{bmatrix}1\\ 0\end{bmatrix}$$

A does not have two (n=2) linearly independent eigenvectors, so *A* is not diagonalizable.

- Steps for diagonalizing an $n \times n$ square matrix:

Step 1: Find n linearly independent eigenvectors $p_1, p_2, \cdots, p_n$

for A with corresponding eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$

Step 2: Let $P = [\, p_1 \mid p_2 \mid \cdots \mid p_n \,]$

Step 3:

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}, \text{ where } Ap_i = \lambda_i p_i,\ i = 1, 2, \ldots, n$$

Note:

The order of the eigenvalues used to form P will determine the order in which the eigenvalues appear on the main diagonal of D.

- Ex 5: (Diagonalizing a matrix)

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

Find a matrix P such that $P^{-1}AP$ is diagonal.

Sol: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda-1 & 1 & 1 \\ -1 & \lambda-3 & -1 \\ 3 & -1 & \lambda+1 \end{vmatrix} = (\lambda-2)(\lambda+2)(\lambda-3) = 0$$

$$\text{Eigenvalues}: \ \lambda_1 = 2, \lambda_2 = -2, \lambda_3 = 3$$

$\lambda_1 = 2$

$$\Rightarrow \lambda_1 \mathrm{I} - A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 3 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \text{ Eigenvector: } p_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda_2 = -2$

$$\Rightarrow \lambda_2 \mathrm{I} - A = \begin{bmatrix} -3 & 1 & 1 \\ -1 & -5 & -1 \\ 3 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}t \\ -\frac{1}{4}t \\ t \end{bmatrix} = \frac{1}{4}t \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \Rightarrow \text{ Eigenvector: } p_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$\lambda_3 = 3$

$$\Rightarrow \lambda_3 \mathrm{I} - A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 3 & -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \text{ Eigenvector}: \ p_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{Let } P = [p_1 \quad p_2 \quad p_3] = \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & 1 \end{bmatrix}$$

$$\Rightarrow \quad P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- Notes: k is a positive integer

(1) $D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \Rightarrow D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$

(2) $D = P^{-1}AP$

$$\begin{aligned} \Rightarrow \quad D^k &= (P^{-1}AP)^k \\ &= (P^{-1}AP)(P^{-1}AP)\cdots(P^{-1}AP) \\ &= P^{-1}A(PP^{-1})A(PP^{-1})\cdots(PP^{-1})AP \\ &= P^{-1}AA\cdots AP \\ &= P^{-1}A^kP \end{aligned}$$

$\therefore A^k = PD^kP^{-1}$

- Thm 7.6: (Sufficient conditions for diagonalization)

  If an $n \times n$ matrix A has n distinct eigenvalues, then the corresponding eigenvectors are linearly independent and A is diagonalizable.

- Ex 7: (Determining whether a matrix is diagonalizable)

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

Sol: Because A is a triangular matrix,

its eigenvalues are the main diagonal entries.

$$\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = -3$$

These three values are distinct, so A is diagonalizable. (Thm.7.6)