5.3 Inner Product

- Inner product:

Let **u**, **v**, and **w** be vectors in a vector space V, and let c be any scalar. An inner product on V is a function that associates a real number <**u**, **v**> with each pair of vectors **u** and **v** and satisfies the following axioms.

(1) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$

(2) $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$

(3) $c \langle \mathbf{u}, \mathbf{v} \rangle = \langle c\mathbf{u}, \mathbf{v} \rangle$

(4) $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = 0$

- Note:

$$\mathbf{u} \cdot \mathbf{v} = \text{dot product (Euclidean inner product for } R^n)$$

$$< \mathbf{u}, \mathbf{v} >= \text{general inner product for vector space } V$$

- Note:

A vector space V with an inner product is called an inner product space.

Vector space: $(V, +, \bullet)$

Inner product space: $(V, +, \bullet, <,>)$

- Ex 1: (The Euclidean inner product for R^n)

Show that the dot product in R^n satisfies the four axioms of an inner product.

Sol:

$$\mathbf{u} = (u_1, u_2, \cdots, u_n) \quad , \quad \mathbf{v} = (v_1, v_2, \cdots, v_n)$$

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

By Theorem 5.3, this dot product satisfies the required four axioms. Thus it is an inner product on R^n.

- Ex 2: (A different inner product for R^n)

Show that the function defines an inner product on R^2, where $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$.

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + 2u_2 v_2$$

Sol:

$$(a) \quad \langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + 2u_2 v_2 = v_1 u_1 + 2v_2 u_2 = \langle \mathbf{v}, \mathbf{u} \rangle$$

$$(b) \quad \mathbf{w} = (w_1, w_2)$$

$$\begin{aligned} \Rightarrow \langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle &= u_1(v_1 + w_1) + 2u_2(v_2 + w_2) \\ &= u_1 v_1 + u_1 w_1 + 2u_2 v_2 + 2u_2 w_2 \\ &= (u_1 v_1 + 2u_2 v_2) + (u_1 w_1 + 2u_2 w_2) \\ &= \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle \end{aligned}$$

$$(c)\quad c\,\langle \mathbf{u}\,,\,\mathbf{v}\rangle = c(u_1v_1 + 2u_2v_2) = (cu_1)v_1 + 2(cu_2)v_2 = \langle c\mathbf{u}\,,\,\mathbf{v}\rangle$$

$$(d)\quad \langle \mathbf{v}\,,\,\mathbf{v}\rangle = v_1{}^2 + 2v_2{}^2 \geq 0$$

$$\langle \mathbf{v}\,,\,\mathbf{v}\rangle = 0 \Rightarrow v_1{}^2 + 2v_2{}^2 = 0 \qquad \Rightarrow \quad v_1 = v_2 = 0 \quad (\mathbf{v} = 0)$$

- Note: (An inner product on R^n)

$$\langle \mathbf{u}\,,\,\mathbf{v}\rangle = c_1u_1v_1 + c_2u_2v_2 + \cdots + c_nu_nv_n\,, \qquad c_i > 0$$

- Ex 3: (A function that is not an inner product)

Show that the following function is not an inner product on R^3.

$$\langle \mathbf{u} \cdot \mathbf{v} \rangle = u_1 v_1 - 2u_2 v_2 + u_3 v_3$$

Sol:

Let $\mathbf{v} = (1, 2, 1)$

Then $< \mathbf{v}, \mathbf{v} >= (1)(1) - 2(2)(2) + (1)(1) = -6 < 0$

Axiom 4 is not satisfied.

Thus this function is not an inner product on R^3.

- Thm 5.7: (Properties of inner products)

  Let **u, v**, and **w** be vectors in an inner product space *V*, and let *c* be any real number.

  (1) $\langle \mathbf{0}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{0} \rangle = 0$

  (2) $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$

  (3) $\langle \mathbf{u}, c\mathbf{v} \rangle = c \langle \mathbf{u}, \mathbf{v} \rangle$

- Norm (length) of **u**:

  $$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$$

- Note:

  $$\|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle$$

- Distance between **u** and **v**:

$$d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \sqrt{\langle \mathbf{u}-\mathbf{v}, \mathbf{u}-\mathbf{v}\rangle}$$

- Angle between two nonzero vectors **u** and **v**:

$$\cos\theta = \frac{\langle \mathbf{u},\mathbf{v}\rangle}{\|\mathbf{u}\|\,\|\mathbf{v}\|},\ 0 \le \theta \le \pi$$

- Orthogonal: $(\mathbf{u} \perp \mathbf{v})$

u and **v** are **orthogonal** if $\langle \mathbf{u},\mathbf{v}\rangle = 0$.

- Notes:

(1) If $\|\mathbf{v}\| = 1$, then v is called a unit vector.

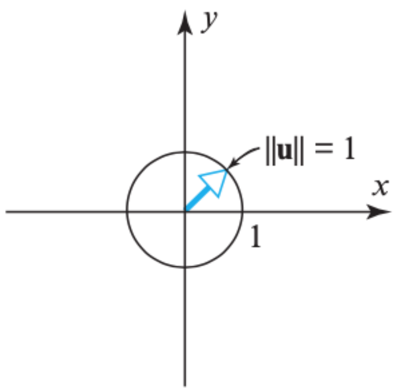
(2) $\|\mathbf{v}\| \neq 1$, $\mathbf{v} \neq 0$ $\xrightarrow{\text{Normalizing}}$ $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ (the unit vector in the direction of **v**)

not a unit vector

DEFINITION 3 If V is an inner product space, then the set of points in V that satisfy

$$\|\mathbf{u}\| = 1$$

is called the ***unit sphere*** or sometimes the ***unit circle*** in V.



(a) The unit circle using the standard Euclidean inner product.

- Ex 6: (Finding inner product)

$$\langle p\ ,\ q\rangle = a_0b_0 + a_1b_1 + \cdots + a_nb_n$$ is an inner product

Let $p(x) = 1 - 2x^2,\ q(x) = 4 - 2x + x^2$ be polynomials in $P_2(x)$

(a) $\langle p\, , q\rangle = ?$ (b) $\|q\| = ?$ (c) $d(p\, , q) = ?$

Sol:

(a) $\langle p\, , q\rangle = (1)(4) + (0)(-2) + (-2)(1) = 2$

(b) $\|q\| = \sqrt{\langle q\, , q\rangle} = \sqrt{4^2 + (-2)^2 + 1^2} = \sqrt{21}$

(c) $\because\ p - q = -3 + 2x - 3x^2$

$$\therefore d(p\, , q) = \|p - q\| = \sqrt{\langle p - q, p - q\rangle}$$

$$= \sqrt{(-3)^2 + 2^2 + (-3)^2} = \sqrt{22}$$

EXAMPLE 6 The Standard Inner Product on M_{nn}

If $\mathbf{u} = U$ and $\mathbf{v} = V$ are matrices in the vector space M_{nn}, then the formula

$$\langle \mathbf{u}, \mathbf{v} \rangle = \operatorname{tr}(U^T V) \tag{8}$$

defines an inner product on M_{nn} called the ***standard inner product*** on that space (see Definition 8 of Section 1.3 for a definition of trace). This can be proved by confirming that the four inner product space axioms are satisfied, but we can see why this is so by computing (8) for the 2×2 matrices

$$U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$$

This yields

$$\langle \mathbf{u}, \mathbf{v} \rangle = \operatorname{tr}(U^T V) = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$$

which is just the dot product of the corresponding entries in the two matrices. And it follows from this that

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{\operatorname{tr}\langle U^T U \rangle} = \sqrt{u_1^2 + u_2^2 + u_3^2 + u_4^2}$$

For example, if

$$\mathbf{u} = U = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = V = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$$

then

$$\langle \mathbf{u}, \mathbf{v} \rangle = \operatorname{tr}(U^T V) = 1(-1) + 2(0) + 3(3) + 4(2) = 16$$

and

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{\operatorname{tr}(U^T U)} = \sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{30}$$

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{\operatorname{tr}(V^T V)} = \sqrt{(-1)^2 + 0^2 + 3^2 + 2^2} = \sqrt{14}$$

- Properties of norm:

(1) $\|\mathbf{u}\| \geq 0$

(2) $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}$

(3) $\|c\mathbf{u}\| = |c|\,\|\mathbf{u}\|$

- Properties of distance:

(1) $d(\mathbf{u}, \mathbf{v}) \geq 0$

(2) $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$

(3) $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$

- Thm 5.8：

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space V.

(1) Cauchy-Schwarz inequality:

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \le \|\mathbf{u}\| \, \|\mathbf{v}\|$$ Theorem 5.4

(2) Triangle inequality:

$$\|\mathbf{u} + \mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$$ Theorem 5.5

(3) Pythagorean theorem :

$\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$ Theorem 5.6

- Orthogonal projections in inner product spaces:

  Let **u** and **v** be two vectors in an inner product space *V*, such that $\mathbf{v} \neq \mathbf{0}$. Then the **orthogonal projection of u onto v** is given by

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\langle \mathbf{u}\, ,\, \mathbf{v} \rangle}{\langle \mathbf{v}\, ,\, \mathbf{v} \rangle}\mathbf{v}$$

- Note:

  If **v** is a init vector, then $\langle \mathbf{v}\, ,\, \mathbf{v} \rangle = \|\mathbf{v}\|^2 = 1$.

  The formula for the orthogonal projection of **u** onto **v** takes the following simpler form.

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \langle \mathbf{u}\, ,\, \mathbf{v} \rangle \mathbf{v}$$

- Ex 10: (Finding an orthogonal projection in R^3)

  Use the Euclidean inner product in R^3 to find the orthogonal projection of u=(6, 2, 4) onto v=(1, 2, 0).

Sol:

$$\because \langle \mathbf{u}, \mathbf{v} \rangle = (6)(1) + (2)(2) + (4)(0) = 10$$

$$\langle \mathbf{v}, \mathbf{v} \rangle = 1^2 + 2^2 + 0^2 = 5$$

$$\therefore \text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v} = \tfrac{10}{5}(1, 2, 0) = (2, 4, 0)$$

- Note:

$\boldsymbol{u} - \text{proj}_v \boldsymbol{u} = (6, 2, 4) - (2, 4, 0) = (4, -2, 4)$ is orthogonal to $v = (1,2,0)$.

- Thm 5.9: (Orthogonal projection and distance)

Let **u** and **v** be two vectors in an inner product space *V*, such that $\mathbf{v} \neq \mathbf{0}$. Then

$$d(\mathbf{u}, \mathrm{proj}_{\mathbf{v}}\mathbf{u}) < d(\mathbf{u}, c\mathbf{v}), \qquad c \neq \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}$$