Math 106
Integral Calculus
Definite Integral

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The Area Problem

How to find the area under the curve of a function $y = f(x)$ from $x = a$ to $x = b$?

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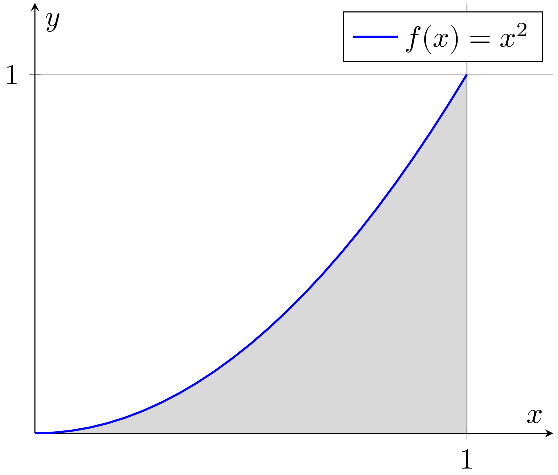

Summation Notation

Summation Notation

Definition

If $\{a_1, a_2, \ldots, a_n\}$ is a set of numbers where n is any positive number , then

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n$$

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Example

Evaluate

$$\sum_{k=1}^{3} (2 + 3k - k^2)$$

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$$\sum_{k=1}^{3} (2 + 3k - k^2)$$

$$\sum_{m=1}^{3} (2 + 3m - m^2)$$

Summation Notation

Theorem

If n is any positive number, c is any real number and $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ are sets of real numbers, then

1. $$\sum_{k=1}^{n} c = nc$$
2. $$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$
3. $$\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$$
4. $$\sum_{k=1}^{n} c a_k = c \left(\sum_{k=1}^{n} a_k \right)$$

Summation Notation

Theorem

$$\sum_{k=1}^{n} k = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Summation Notation

Example

Evaluate

1. $\displaystyle\sum_{k=1}^{100} k$
2. $\displaystyle\sum_{k=1}^{10} k^2$
3. $\displaystyle\sum_{k=1}^{n} (k^2 + 2k - 3)$

Exam Problem

Exam problem

Find the value of the constant c such that

$$\sum_{k=1}^{10}(k^2 + 3c) = 445$$

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$c = 2$

Exam Problem

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Find the value of a so that $\displaystyle\sum_{k=1}^{10}(k^2 - ak) = 0$

Exam Problem

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Find the value of a so that $\sum_{k=1}^{10}(k^2 - ak) = 0$

$a = 7$

Riemann Sum

Riemann sum

Defining the Area with Rectangles

1. Divide $[a, b]$ into n equal subintervals.
2. The width of each subinterval is $\Delta x = \frac{b-a}{n}$.
3. Choose the numbers $x_0, x_1, \dots, x_n$ such that

$$\begin{aligned} x_0 &= a \\ x_1 &= a + \Delta x \\ x_2 &= a + 2\Delta x \\ &\vdots \\ x_n &= a + n\Delta x = b \end{aligned}$$

The set $P = \{x_0, x_1, \dots, x_n\}$ is a partition of the interval $[a, b]$.

Riemann Sum and its limit

Definition

Let f be defined on $[a, b]$, and let P be a partition of $[a, b]$. A Riemann sum of f for P is

$$R_P = \sum_{k=1}^{n} f(w_k)\Delta x$$

where $w_k \in [x_{k-1}, x_k]$

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If $f : [a, b] \to \mathbb{R}$ is continuous and nonnegative, then the area under the curve of f is the limit of Riemann sum

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(w_k)\Delta x$$

Different Choices for w_k

- **Right Endpoint:** $w_k = x_k = a + k\Delta x$.
- **Left Endpoint:** $w_k = x_{k-1} = a + (k-1)\Delta x$.
- **Midpoint:** $w_k = \frac{x_{k-1}+x_k}{2}$

Example

Use Riemann sum to find the area under the curve of $f(x) = x + 2$ in the interval $[0, 3]$.

Definite integral

Definite integral

Definition

Let f be defined on $[a, b]$. The indefinite integral of f from a to b is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(w_k)\Delta x$$

provided the limit exists.

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provided the limit exists.

If the limit exists, then f is integrable on $[a, b]$.

Exam Problem

Exam problem

Use Riemann sum to find $\int_0^2 (x^2+4)\,dx$

Area under the graph of f

Theorem

If f is integrable on $[a,b]$, and

$$f(x) \geq 0, \qquad \forall x \in [a,b]$$

then the area A of the region under the graph of f from a to b is

$$A = \int_a^b f(x)dx$$

Integrable Functions

Theorem

If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

Integrable Functions

Theorem

If f and g are integrable on $[a, b]$ and $c \in \mathbb{R}$, then cf, $f + g$ and $f - g$ are integrable on $[a, b]$ and

1. $$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$
2. $$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$
3. $$\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

Properties of definite integral

Theorem

1. *If* $c \in \mathbb{R}$, *then* $\displaystyle\int_a^b c\,dx = c(b-a)$
2. *If* $f(a)$ *exists, then* $\displaystyle\int_a^a f(x)dx = 0$
3. $\displaystyle\int_a^b f(x)dx = -\int_b^a f(x)dx$

Integrable Functions

Theorem

If $a < c < b$, *and* f *is integrable on* $[a, c]$ *and* $[c, b]$ *then* f *is integrable on* $[a, b]$ *and*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Properties of definite integral

Theorem

If f is integrable on $[a, b]$, and

$$f(x) \geq 0, \qquad \forall x \in [a, b]$$

then

$$\int_a^b f(x)\, dx \geq 0$$

Theorem

If f and g are integrable on $[a, b]$, and

$$f(x) \geq g(x), \qquad \forall x \in [a, b]$$

then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

Properties of definite integral

Example

Evaluate

1. $$\int_{5}^{5} (x+1)\,dx$$
2. $$\int_{-3}^{5} 4\,dx$$
3. $$\int_{1}^{9} f(x)\,dx - \int_{4}^{9} f(x)\,dx$$

Properties of definite integral

Example

Show that

$$\int_{-1}^{2} (x^2 + x + 1)\,dx \geq \int_{-1}^{2} (x - 1)\,dx$$

The fundamental Theorem of Calculus

The fundamental Theorem of Calculus

The fundamental Theorem of Calculus

Theorem

Suppose that f is continuous on $[a, b]$.

1. *If*
$$F(x) = \int_a^x f(t)dt$$
for every $x \in [a, b]$, then F is an antiderivative of f on $[a, b]$.
2. *If F is an antiderivative of f on $[a, b]$, then*
$$\int_a^b f(x)dx = F(b) - F(a)$$

The fundamental Theorem of Calculus

Corollary

If f is continuous on $[a, b]$, and F is any antiderivative of f, then

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

The fundamental Theorem of Calculus

Example

Evaluate

$$\int_{-2}^{2} (3x^2 + 2x)dx$$

The fundamental Theorem of Calculus

Example

Evaluate

$$\int_0^{\pi} \sin x dx$$

The fundamental Theorem of Calculus

Example

Evaluate

$$\int_1^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right) dx$$

The fundamental Theorem of Calculus

Example

Evaluate

$$\int_0^{\frac{\pi}{4}} (\sec^2 x + 2)dx$$

Example

Evaluate

$$\int_0^3 |x-2|\,dx$$

Substitution

Theorem

If $u = g(x)$, *then*

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)\,du$$

The fundamental Theorem of Calculus

Example

Evaluate

$$\int_{2}^{10} \frac{10}{\sqrt{5x-1}}\,dx$$

The fundamental Theorem of Calculus

Example

Evaluate

$$\int_0^{\frac{\pi}{4}} (1 + \sin 2x)^2 \cos 2x \, dx$$

Even and Odd Functions

Even and Odd Functions

Theorem

Lef f be continuous of $[-a, a]$.

1. *If f is an even fucntion,*

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

2. *If f is an odd fucntion,*

$$\int_{-a}^{a} f(x)dx = 0$$

Even and Odd Functions

Example

Evaluate

$$\int_{-2}^{2} (x^3 + 2x)\, dx$$

Example

Evaluate

$$\int_{-1}^{1} (5x^4 + 6x^2)\, dx$$

Even and Odd Functions

Example

Evaluate
$$\int_{-2}^{2} (5x^3 + 6x^2 + 9x)\, dx$$

The fundamental Theorem of Calculus

Theorem

Let f be continuous on $[a, b]$. If $a \leq c \leq b$, then for every $x \in [a, b]$,

$$\frac{d}{dx}\int_c^x f(t)dt = f(x)$$

The fundamental Theorem of Calculus

Example

Evaluate

$$\frac{d}{dx}\int_1^x (t^2+1)\,dt$$

The fundamental Theorem of Calculus

Theorem

Let f be continuous on $[a,b]$. If g and h are in the domain of f and differentiable, then then for every $x \in [a,b]$,

1. $$\frac{d}{dx}\int_a^{h(x)} f(t)dt = f(h(x))h'(x)$$

2. $$\frac{d}{dx}\int_{g(x)}^{b} f(t)dt = -f(g(x))g'(x)$$

The fundamental Theorem of Calculus

Theorem

Let f be continuous on $[a, b]$. If g and h are in the domain of f and differentiable, then then for every $x \in [a, b]$,

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt = f(h(x))h'(x) - f(g(x))g'(x)$$

The fundamental Theorem of Calculus

Example

Evaluate

$$\frac{d}{dx}\int_1^{x^2} \sqrt{\cos^2 t}\, dt$$

The fundamental Theorem of Calculus

Example

Evaluate

$$\frac{d}{dx}\int_{\sin x}^{2} (t^3+2)\,dt$$

Exam Problem

Exam problem

If $F(x) = x^3 \int_0^x \sqrt{2 + \sin t}\, dt$. Find $F'(0)$

Exam Problem

Exam problem

If $F(x) = \displaystyle\int_{2x}^{x^3} \sqrt{1+t^4}\, dt$, find $F'(x)$

Exam Problem

Exam problem

If

$$F(x) = \int_{\cos x}^{\sin x} \sqrt{1 - t^2}\, dt$$

Find $F'(\frac{\pi}{2})$

Mean Value Theorem for integrals

Mean Value Theorem for integrals

Theorem

If f is continuous on $[a, b]$, then there is $c \in (a, b)$, such that

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

Mean Value Theorem for integrals

Example

Find the number c that satisfies the mean value theorem for the function $f(x) = x^2$ on $[0, 3]$.

Exam Problem

Exam problem

Find the number c in the mean value theorem for $f(x) = \dfrac{x}{\sqrt{x^2+9}}$ on $[0, 4]$