Math 106
Integral Calculus
Definite Integral

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Definite Integral

1. Summation Notation
2. Riemann Sum
3. Definite integral
4. The fundamental Theorem of Calculus
5. Mean Value Theorem for integrals

The Area Problem

How to find the area under the curve of a function $y = f(x)$ from $x = a$ to $x = b$?

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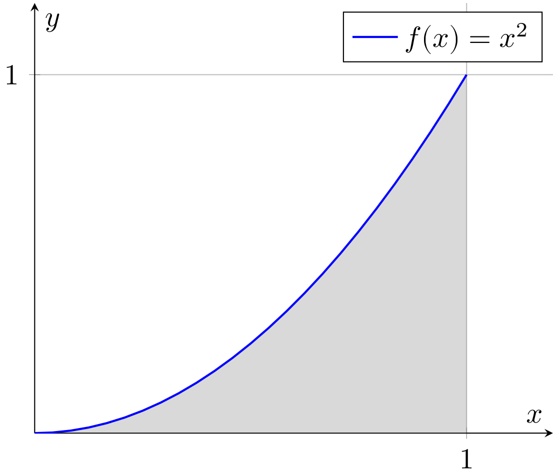

Summation Notation

Summation Notation

Definition

If $\{a_1, a_2, \ldots, a_n\}$ is a set of numbers where n is any positive number , then

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n$$

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Example

Evaluate

$$\sum_{k=1}^{3} (2 + 3k - k^2)$$

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$$\sum_{m=1}^{3} (2 + 3m - m^2)$$

Summation Notation

Theorem

If n *is any positive number,* c *is any real number and* $\{a_1, a_2, \dots, a_n\}$ *and* $\{b_1, b_2, \dots, b_n\}$ *are sets of real numbers, then*

1. $\displaystyle\sum_{k=1}^{n} c = nc$
2. $\displaystyle\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$
3. $\displaystyle\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$
4. $\displaystyle\sum_{k=1}^{n} c a_k = c\left(\sum_{k=1}^{n} a_k\right)$

Summation Notation

Theorem

$$\sum_{k=1}^{n} k = 1 + 2 + ... + n = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + ... + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Summation Notation

Example

Evaluate

1. $$\sum_{k=1}^{100} k$$
2. $$\sum_{k=1}^{10} k^2$$
3. $$\sum_{k=1}^{n} (k^2 + 2k - 3)$$

Exam Problems

Example

Find the value of the constant c such that

$$\sum_{k=1}^{10}(k^2 + 3c) = 445$$

Exam Problems

Example

Find the value of the constant c such that

$$\sum_{k=1}^{10}(k^2 + 3c) = 445$$

$c = 2$

Exam Problems

Example

Find the value of a so that $\sum_{k=1}^{10}(k^2 - ak) = 0$

Exam Problems

Example

Find the value of a so that $\sum_{k=1}^{10}(k^2 - ak) = 0$

$a = 7$

Riemann Sum

Riemann sum

Defining the Area with Rectangles

1. Divide $[a, b]$ into n equal subintervals.
2. The width of each subinterval is $\Delta x = \frac{b-a}{n}$.
3. Choose the numbers $x_0, x_1, \dots, x_n$ such that

$$\begin{aligned} x_0 &= a \\ x_1 &= a + \Delta x \\ x_2 &= a + 2\Delta x \\ &\vdots \\ x_n &= a + n\Delta x = b \end{aligned}$$

The set $P = \{x_0, x_1, \dots, x_n\}$ is a partition of the interval $[a, b]$.

Riemann Sum and its limit

Definition

Let f be defined on $[a, b]$, and let P be a partition of $[a, b]$. A Riemann sum of f for P is

$$R_P = \sum_{k=1}^{n} f(w_k)\Delta x$$

where $w_k \in [x_{k-1}, x_k]$

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If $f : [a, b] \to \mathbb{R}$ is continuous and nonnegative, then the area under the curve of f is the limit of Riemann sum

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(w_k)\Delta x$$

Different Choices for w_k

- **Right Endpoint:** $w_k = x_k = a + k\Delta x$.
- **Left Endpoint:** $w_k = x_{k-1} = a + (k-1)\Delta x$.
- **Midpoint:** $w_k = \frac{x_{k-1}+x_k}{2}$

Reimann sum

Example

Use Riemann sum to find the area under the curve of $f(x) = x + 2$ in the interval $[0, 3]$.

Definite integral

Definite integral

Definition

Let f be defined on $[a, b]$. The indefinite integral of f from a to b is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(w_k)\Delta x$$

provided the limit exists.

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provided the limit exists.

If the limit exists, then f is integrable on $[a, b]$.

Exam Problem

Example

Use Riemann sum to find $\int_0^2 (x^2 + 4)\, dx$

Area under the graph of f

Theorem

If f is integrable on $[a, b]$, and

$$f(x) \geq 0, \qquad \forall x \in [a, b]$$

then the area A of the region under the graph of f from a to b is

$$A = \int_a^b f(x)dx$$

Integrable Functions

Theorem

If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

Integrable Functions

Theorem

If f *and* g *are integrable on* $[a, b]$ *and* $c \in \mathbb{R}$*, then* cf*,* $f + g$ *and* $f - g$ *are integrable on* $[a, b]$ *and*

1. $$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$
2. $$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$
3. $$\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

Properties of definite integral

Theorem

1. *If* $c \in \mathbb{R}$, *then* $\displaystyle\int_a^b c\,dx = c(b-a)$
2. *If* $f(a)$ *exists, then* $\displaystyle\int_a^a f(x)dx = 0$
3. $\displaystyle\int_a^b f(x)dx = -\int_b^a f(x)dx$

Integrable Functions

Theorem

If $a < c < b$*, and* f *is integrable on* $[a, c]$ *and* $[c, b]$ *then* f *is integrable on* $[a, b]$ *and*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Properties of definite integral

Theorem

If f *is integrable on* $[a, b]$, *and*

$$f(x) \geq 0, \qquad \forall x \in [a, b]$$

then

$$\int_a^b f(x)\, dx \geq 0$$

Properties of definite integral

Theorem

If f *and* g *are integrable on* $[a, b]$, *and*

$$f(x) \geq g(x), \qquad \forall x \in [a, b]$$

then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

Example

Evaluate

1. $$\int_5^5 (x+1)\,dx$$
2. $$\int_{-3}^5 4\,dx$$
3. $$\int_1^9 f(x)\,dx - \int_4^9 f(x)\,dx$$

Properties of definite integral

Example

Show that

$$\int_{-1}^{2} (x^2 + x + 1)\, dx \geq \int_{-1}^{2} (x - 1)\, dx$$

The fundamental Theorem of Calculus

The fundamental Theorem of Calculus

Theorem

Suppose that f *is continuous on* $[a,b]$.

1. *If*

$$F(x)=\int_a^x f(t)dt$$

for every $x\in[a,b]$, *then* F *is an antiderivative of* f *on* $[a,b]$.

2. *If* F *is an antiderivative of* f *on* $[a,b]$, *then*

$$\int_a^b f(x)dx=F(b)-F(a)$$

The fundamental Theorem of Calculus

Corollary

Ift f is continuous on $[a, b]$, and F is any antiderivative of f, then

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

The fundamental Theorem of Calculus

Example

Evaluate

$$\int_{-2}^{2} (3x^2 + 2x)dx$$

Example

Evaluate

$$\int_0^{\pi} \sin x dx$$

The fundamental Theorem of Calculus

Example

Evaluate

$$\int_1^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$$

Example

Evaluate

$$\int_0^{\frac{\pi}{4}} (\sec^2 x + 2)dx$$

The fundamental Theorem of Calculus

Example

Evaluate

$$\int_0^3 |x-2|\,dx$$

Substitution

Theorem

If $u = g(x)$, *then*

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)\, du$$

Example

Evaluate
$$\int_{2}^{10} \frac{10}{\sqrt{5x-1}}\,dx$$

Example

Evaluate

$$\int_0^{\frac{\pi}{4}} (1+\sin 2x)^2 \cos 2x \, dx$$

Even and Odd Functions

Even and Odd Functions

Theorem

Lef f be continuous of $[-a, a]$.

1. *If f is an even fucntion,*

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

2. *If f is an odd fucntion,*

$$\int_{-a}^{a} f(x)dx = 0$$

Even and Odd Functions

Example

Evaluate

$$\int_{-2}^{2} (x^3 + 2x)\, dx$$

Example

Evaluate

$$\int_{-1}^{1} (5x^4 + 6x^2)\, dx$$

Even and Odd Functions

Example

Evaluate

$$\int_{-2}^{2} (5x^3 + 6x^2 + 9x)\, dx$$

The fundamental Theorem of Calculus

Theorem

Let f *be continuous on* $[a, b]$. *If* $a \leq c \leq b$, *then for every* $x \in [a, b]$,

$$\frac{d}{dx} \int_c^x f(t)dt = f(x)$$

The fundamental Theorem of Calculus

Example

Evaluate

$$\frac{d}{dx}\int_1^x (t^2+1)\,dt$$

Example

Evaluate

$$\frac{d}{dx}\int_1^3 \sqrt{t^2+4t}\,dt$$

The fundamental Theorem of Calculus

Theorem

Let f be continuous on $[a,b]$. If g and h are in the domain of f and differentiable, then then for every $x \in [a,b]$,

1. $$\frac{d}{dx}\int_{a}^{h(x)} f(t)dt = f(h(x))h'(x)$$

2. $$\frac{d}{dx}\int_{g(x)}^{b} f(t)dt = -f(g(x))g'(x)$$

The fundamental Theorem of Calculus

Theorem

Let f be continuous on $[a, b]$. If g and h are in the domain of f and differentiable, then then for every $x \in [a, b]$,

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t)dt = f(h(x))h'(x) - f(g(x))g'(x)$$

Example

Evaluate

$$\frac{d}{dx}\int_{1}^{x^2}\sqrt{\cos^2 t}\,dt$$

The fundamental Theorem of Calculus

Example

Evaluate

$$\frac{d}{dx}\int_{\sin x}^{2}(t^3+2)\,dt$$

Exam Problem

Example

If $F(x) = x^3 \int_0^x \sqrt{2 + \sin t}\, dt$. Find $F'(0)$

Exam Problem

Example

If $F(x) = \int_{2x}^{x^3} \sqrt{1 + t^4}\, dt$, find $F'(x)$

Exam Problem

Example

If

$$F(x) = \frac{d}{dx} \int_{\cos x}^{\sin x} \sqrt{1-t^2}\, dt$$

Find $F'(\frac{\pi}{2})$

Mean Value Theorem for integrals

Mean Value Theorem for integrals

Theorem

If f *is continuous on* $[a, b]$*, then there is* $c \in (a, b)$*, such that*

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

Example

Find the number c that satisfies the mean value theorem for the function $f(x) = x^2$ on $[0, 3]$.

Exam problem

Example

Find the number c in the mean value theorem for $f(x) = \dfrac{x}{\sqrt{x^2+9}}$ on $[0, 4]$