Math 204
Differential Equations
First Order Differential Equations

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First Order Differential Equations

First Order Differential Equations

Consider the equation of order one

$$F(x, y, y') = 0$$

We suppose that the equation can be written as

$$y' = \frac{dy}{dx} = f(x, y)$$

This equation can be written in the form

$$M(x, y)dx + N(x, y)dy = 0$$

Initial Value Problems

Initial value problems

$$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)})$$

$$y(x_0) = y_0$$

$$y'(x_0) = y_1$$

$$\dots$$

$$y^{(n-1)}(x_0) = y_{n-1}$$

Initial value problems

First order

$$\frac{dy}{dx} = f(x, y)$$

$$y(x_0) = y_0$$

Second order

$$\frac{d^2y}{dx^2} = f(x, y, y')$$

$$y(x_0) = y_0$$

$$y'(x_0) = y_1$$

Differentiation

Example

Find the derivative

$$xy = 4$$

Initial value problems

Example

Solve

$$y' = x - 3$$

$$y(0) = 1$$

Initial value problems

Example

Solve

$$y' = y$$

$$y(0) = 4$$

Example

$$y' = xy^{\frac{1}{2}}$$

$$y(0) = 0$$

This equation has the solutions

$$y = 0$$

and

$$y = \frac{x^4}{16}$$

Existence of a unique solution

Existence of a unique solution

Theorem

Consider a first order differential equation

$$\frac{dy}{dx} = f(x, y)$$

with initial condition

$$y(x_0) = y_0$$

Let R *be a rectangular region defined by* $a \leq x \leq b, c \leq y \leq d$ *that contains the point* (x_0, y_0) *in its interior.*
If f *and* $\frac{\partial f}{\partial y}$ *are continuous on* R *then there exists an interval* I *centered at* x_0 *and a unique function* $y(x)$ *satisfies the IVP.*

Existence of a unique solution

Example

Find the largest region for which the IVP has a unique solution

$$\sqrt{x^2 - 4}\, y' = (1 + \sin x) \ln y$$

$$y(3) = 4$$

Existence of a unique solution

Example

Find the largest region for which the IVP has a unique solution

$$\sqrt{x^2-4}\; y' = (1+\sin x)\ln y$$

$$y(3) = 4$$

$$\{(x,y) : x > 2, y > 0\}$$

Existence of a unique solution

Example

Find the largest region for which the IVP has a unique solution

$$\ln(x-2)\; y' = \sqrt{y-2}$$

$$y\left(\frac{5}{2}\right) = 5$$

Existence of a unique solution

Example

Find the largest region for which the IVP has a unique solution

$$\ln(x-2)\ y' = \sqrt{y-2}$$

$$y\left(\frac{5}{2}\right) = 5$$

$$\{(x,y) : 2 < x < 3, y > 2\}$$

Existence of a unique solution

Example

Find the largest region for which the IVP has a unique solution

$$\sqrt{\frac{x}{y}}\, y' = \cos(x+y), \qquad y \neq 0$$

$$y\,(1) = 1$$

Existence of a unique solution

Example

Find the largest region for which the IVP has a unique solution

$$\sqrt{\frac{x}{y}}\, y' = \cos(x+y), \qquad y \neq 0$$

$$y\,(1) = 1$$

$$\{(x,y) : x > 0, y > 0\}$$

Separable Equations

Definition

Consider a first-order differential equation of the form

$$M(x,y)dx + N(x,y)dy = 0$$

If we can write it in the form

$$g(x)dx = h(y)dy$$

then it is said to be separable.

To solve separable equations, we integrate each part

$$\int g(x)dx = \int h(y)dy + c$$

Separable equations

Example

Solve the differential equation

$$\frac{dy}{dx} = x^2$$

Separable equations

Example

Solve the differential equation

$$\frac{dy}{dx} = y$$

Separable equations

Example

Solve the differential equation

$$\frac{dy}{dx} = xy$$

Separable equations

Example

Solve the differential equation

$$2x(y^2 + y)dx + (x^2 - 1)ydy = 0, \qquad y \neq 0$$

Separable equations

Example

Solve the differential equation

$$(xy + x)dx = (x^2y^2 + x^2 + y^2 + 1)dy$$

Example

Solve the Initial value differential equation

$$e^y \frac{dy}{dx} = \cos(2x) + 2e^y \sin^2(x) - 1$$

$$y(\frac{\pi}{2}) = \ln 2$$

Separable equations

Example

Solve the Initial value differential equation

$$e^y \frac{dy}{dx} = \cos(2x) + 2e^y \sin^2(x) - 1$$

$$y(\frac{\pi}{2}) = \ln 2$$

$$\ln|e^y - 1| + \frac{\sin 2x}{2} + \frac{\pi}{2} - x = 0$$

Equations with Homogeneous Coefficients

Equations with Homogeneous Coefficients

Definition

A function $f(x, y)$ is called homogeneous of degree n, if

$$f(tx, ty) = t^n f(x, y), \qquad \forall\, t > 0$$

Homogeneous function

Example

$$f(x,y) = x^2 + 2xy - 2y^2$$

Homogeneous function

Example

$$f(x, y) = x^3y + y^3x + 2xy$$

Homogeneous function

Example

$$f(x, y) = x - 3y + \sqrt{x^2 + 2y^2}$$

Homogeneous function

Example

$$f(x, y) = x \ln x - x \ln y$$

Homogeneous function

Example

$$M(x,y) = x^2 - y^2, \qquad N(x,y) = x^2 + 2y^2$$

Homogeneous function

Example

$$M(x,y) = x^2 - y^2, \qquad N(x,y) = x^2 + 2y^2$$

$$f(x,y) = \frac{M(x,y)}{N(x,y)}$$

Equations with Homogeneous Coefficients

Definition

A differential equation of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is called **an equation with homogeneous coefficients** if both $M(x,y)$ and $N(x,y)$ are homogeneous functions of the same degree.

Solution of Equations with Homogeneous Coefficients

To solve an equation with homogeneous coefficients, we use the substitution

$$y = ux$$

or

$$x = vy$$

This will reduce the equation to a separable first order DE.

Example

Solve

$$(x^2 + y^2)dx + (x^2 - xy)dy = 0$$

Example

Solve

$$(x^2 + y^2)dx + (x^2 - xy)dy = 0$$

Solution:

$$(x + y)^2 = ce^{\frac{y}{x}}$$

Equations with Homogeneous Coefficients

Example

Solve

$$\frac{dy}{dx} + \frac{3xy + y^2}{x^2 + xy} = 0, \qquad x \neq 0, \quad y \neq -x$$

Example

Solve

$$\frac{dy}{dx} + \frac{3xy + y^2}{x^2 + xy} = 0, \qquad x \neq 0, \quad y \neq -x$$

Solution:

$$yx^2(2x + y) = c_1$$

Equations with Homogeneous Coefficients

Example

Solve
$$y\,dx + x\left(\ln\frac{x}{y} - 1\right)dy = 0, \qquad y(1) = e$$

Example

Solve

$$ydx + x\left(\ln\frac{x}{y} - 1\right)dy = 0, \qquad y(1) = e$$

Solution:

$$y\ln\frac{x}{y} = -e, \qquad x > 0, y > 0$$

Appropriate Substitution

Appropriate Substitution

If we have a differential equation of the form

$$\frac{dy}{dx} = f(ax + by)$$

we use the substitution

$$u = ax + by$$

then

$$\frac{du}{dx} = a + b\frac{dy}{dx}$$

Appropriate Substitution

Example

Solve
$$\frac{dy}{dx} = (-2x + y)^2 - 7$$

Appropriate Substitution

Example

Solve

$$\frac{dy}{dx} = (-2x + y)^2 - 7$$

Solution:

$$\ln\left|\frac{-2x + y - 3}{-2x + y + 3}\right| - 6x = c$$

Equations with linear coefficients

Consider the differential equation

$$(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0$$

where $a_1, b_1, c_1, a_2, b_2, c_2$ are real constants.

The two lines $a_1x + b_1y + c_1 = 0$, and $a_2x + b_2y + c_2 = 0$ are parallel, or intersected.

Equations with linear coefficients

Example

Solve

$$\frac{dy}{dx} = \frac{1 - 4x - 4y}{x + y}, \qquad x + y \neq 0$$

Equations with linear coefficients

Example

Solve

$$\frac{dy}{dx} = \frac{1-4x-4y}{x+y}, \qquad x+y \neq 0$$

Solution:

$$\frac{x+y}{3} + \frac{1}{9}\ln|1-3x-3y| + x = c$$

Equations with linear coefficients

Example

Solve

$$\frac{dy}{dx} = \frac{x-y-3}{x+y-1}, \qquad x+y-1 \neq 0$$

Example

Solve

$$\frac{dy}{dx} = \frac{x-y-3}{x+y-1}, \qquad x+y-1 \neq 0$$

Solution:

$$(x-2)^2 - 2(x-2)(y+1) - (y+1)^2 = c_1$$

Appropriate Substitution

Example

Use the substitution $u = xy$ to solve

$$\frac{dy}{dx} = \frac{y(1+xy)}{x(1-xy)}, \qquad x > 0, y > 0, xy \neq 1$$

Appropriate Substitution

Example

Use the substitution $u = xy$ to solve

$$\frac{dy}{dx} = \frac{y(1+xy)}{x(1-xy)}, \qquad x > 0, y > 0, xy \neq 1$$

Solution:

$$\frac{y}{x} = e^{xy} c_1, \qquad c_1 \neq 0$$