UNIT 1B

Matrices and Matrix Algebra

Unit 1B · Sections 1.3 – 1.4

MATH 244 — Linear Algebra
Elementary Linear Algebra, Anton & Rorres, 11th edition
King Saud University · Department of Mathematics

What you should be able to do

Section 1.3

- State the size of a matrix and read off the entry $(A)_{ij} = a_{ij}$.
- Add, subtract and scalar-multiply matrices, and form linear combinations.
- Decide when a product is defined, give its size, and compute it by the row-column rule.
- Compute a product by columns, by rows, and by column-row expansion.
- Write a linear system as $A\mathbf{x} = \mathbf{b}$ and move between all four representations.
- Compute a transpose and a trace.

Section 1.4

- Use the algebraic properties of matrix arithmetic and know which familiar rules FAIL.
- Decide whether a 2×2 matrix is invertible and compute its inverse.
- Solve $A\mathbf{x} = \mathbf{b}$ by $\mathbf{x} = A^{-1}\mathbf{b}$ when A is invertible.
- Use $(AB)^{-1} = B^{-1}A^{-1}$, $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$ and the rules for powers.

SECTION 1.3

Matrices and Matrix Operations

GOAL — Stop treating a matrix as shorthand for a system and start treating it as an object with its own arithmetic.

1. Matrix notation, size and entries
2. Equality, addition, scalar multiplication
3. Matrix multiplication and the row-column rule
4. Products by columns, by rows, and column-row expansion
5. The matrix form $Ax = b$
6. Transpose and trace

DEFINITION

Matrix, entries, size

A MATRIX is a rectangular array of numbers. The numbers in the array are the ENTRIES of the matrix.

- The SIZE of a matrix is the number of rows (horizontal) by the number of columns (vertical). Rows always come first: a 3×2 matrix has three rows and two columns.

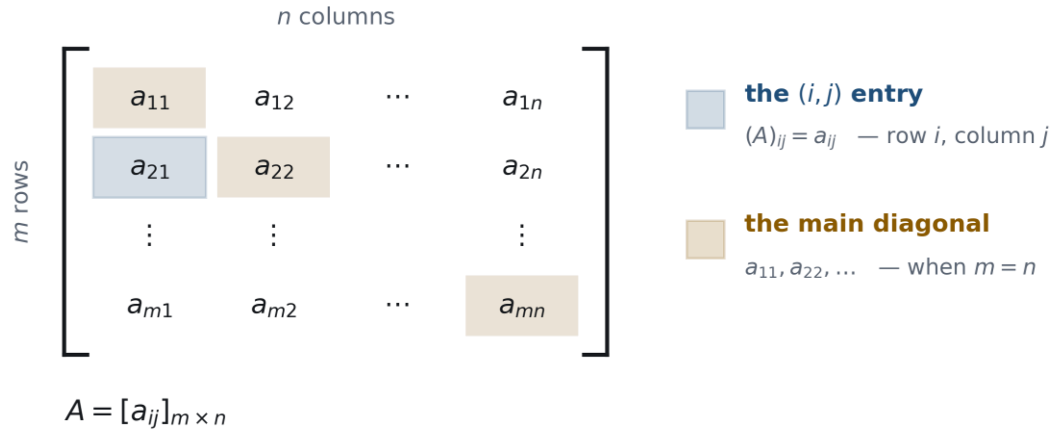
Example 1 — matrices of every shape

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 & 0 & -3 \end{bmatrix} \quad \begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & 1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 4 \end{bmatrix}$$

- Their sizes, left to right, are 3×2, 1×4, 3×3, 2×1 and 1×1.
- A matrix with one row is a ROW VECTOR; with one column, a COLUMN VECTOR. The 1×1 matrix $[4]$ is both.

NOTATION

Naming the parts of a matrix



- A matrix with n rows and n columns is a SQUARE MATRIX OF ORDER n; the entries $a_{11}, a_{22}, \ldots, a_{nn}$ lie on its MAIN DIAGONAL.
- Capital letters denote matrices, lower-case letters denote scalars, and bold lower-case letters denote row or column vectors: $\mathbf{a} = [a_1 \quad \cdots \quad a_n]$.

DEFINITION

Equality of matrices

> Two matrices are EQUAL if they have the same size and their corresponding entries are equal: $(A)_{ij} = (B)_{ij}$ for every i and j.

Example 2

$$A = \begin{bmatrix} 2 & 1 \\ 3 & x \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \end{bmatrix}$$

- If $x = 5$ then $A = B$; for every other value of x they are not equal.
- There is NO value of x making $A = C$ — they have different sizes, so the question of matching entries never arises.

DEFINITION

Addition, subtraction, scalar multiplication

- If A and B have the SAME SIZE, then $A+B$ and $A-B$ are formed entrywise. Matrices of different sizes cannot be added or subtracted.

$$(A+B)_{ij} = (A)_{ij} + (B)_{ij}, \qquad (A-B)_{ij} = (A)_{ij} - (B)_{ij}$$

- If c is a scalar, the SCALAR MULTIPLE cA multiplies every entry by c: $(cA)_{ij} = c(A)_{ij}$. We write $-B$ for $(-1)B$, so that $A-B = A+(-1)B$.

Example 3

$$A = \begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 4 & -2 & 7 & 0 \end{bmatrix} \qquad A+B = \begin{bmatrix} -2 & 4 & 5 & 4 \\ 1 & 2 & 2 & 3 \\ 7 & 0 & 3 & 5 \end{bmatrix}$$

$$B = \begin{bmatrix} -4 & 3 & 5 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & 2 & -4 & 5 \end{bmatrix} \qquad A-B = \begin{bmatrix} 6 & -2 & -5 & 2 \\ -3 & -2 & 2 & 5 \\ 1 & -4 & 11 & -5 \end{bmatrix}$$

EXAMPLE

Linear combinations of matrices

A LINEAR COMBINATION of matrices of the same size is a sum of scalar multiples of them. Rewrite every subtraction as $+(-1)\times$ before distributing.

$$A_1 = \begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 3 & 1 & 1 \\ -2 & -1 & 7 \end{bmatrix}$$

$$2A_1 - A_2 + A_3 = \begin{bmatrix} 2 & 4 & 2 \\ 6 & -2 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 1 \\ -2 & -1 & 7 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 2 \\ 4 & -4 & 6 \end{bmatrix}$$

- Every entry is handled independently, so a linear combination of $m \times n$ matrices is again $m \times n$.

DEFINITION

Matrix multiplication — first, a row times a column

Multiplication is NOT entry wise. It is built from a row matrix times a column matrix OF THE SAME LENGTH.

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_r \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_r \end{bmatrix} = [y_1 z_1 + y_2 z_2 + \cdots + y_r z_r]$$

Example

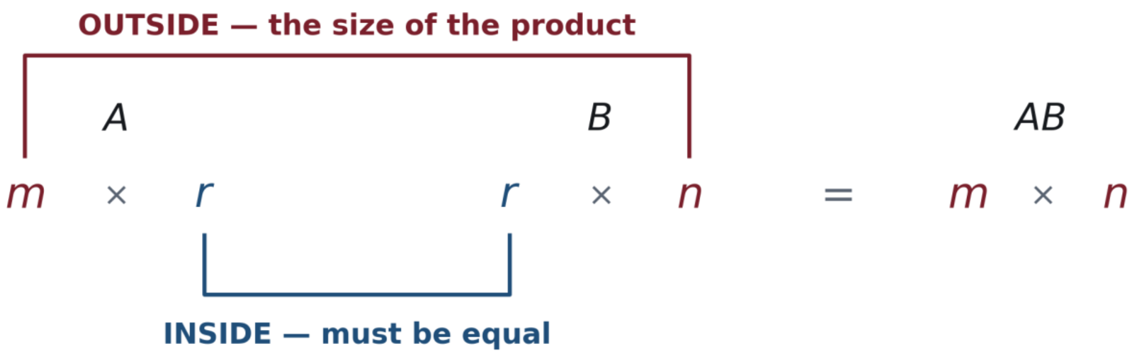
$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = [(1)(-1) + (2)(0) + (3)(1)] = [2]$$

- The answer is a 1×1 MATRIX. Every entry of a general product will be one of these numbers.

DEFINITION

The product AB, and when it exists

If A is $m \times r$ and B is $r \times n$, then AB is the $m \times n$ matrix whose (i, j) entry is row i of A times column j of B.



- Write the two sizes side by side. If the INSIDE numbers agree the product is defined, and the OUTSIDE numbers give its size.

EXAMPLE 6

Is the product defined?

Suppose A, B and C have sizes

$$\underset{3\times 4}{A} \qquad \underset{4\times 7}{B} \qquad \underset{7\times 3}{C}$$

Defined

- AB is 3×7
- BC is 4×3
- CA is 7×4

Undefined

- AC — inside numbers 4 and 7
- CB — inside numbers 3 and 4
- BA — inside numbers 7 and 3

- Notice that AB exists while BA does not. Order matters before any arithmetic happens.

EXAMPLE 5

Computing one entry of a product

column 3 of B

$$\begin{bmatrix}1&2&4\\2&6&0\end{bmatrix}\begin{bmatrix}4&1&4&3\\0&-1&3&1\\2&7&5&2\end{bmatrix}=\begin{bmatrix}\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\boxed{26}&\cdot\end{bmatrix}$$

row 2 of A

$(AB)_{23} = (2)(4) + (6)(3) + (0)(5) = 26$

- Cover everything except row i of A and column j of B; pair the entries in order, multiply, and add.
- Repeating this for every position gives $AB = \begin{bmatrix}12 & 27 & 30 & 13\\ 8 & -4 & 26 & 12\end{bmatrix}$.

The row-column rule

In general, if $A = [a_{ij}]$ is $m \times r$ and $B = [b_{ij}]$ is $r \times n$, then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{ir}b_{rj}$$

- The first subscript on the a's is fixed at i and the second runs $1,2,\ldots,r$: we are walking ACROSS row i.
- The second subscript on the b's is fixed at j and the first runs $1,2,\ldots,r$: we are walking DOWN column j.
- The two walks have the same length r — which is exactly the condition that made the product defined.

TECHNIQUE

Products by columns and by rows

A product can be assembled a whole column or a whole row at a time — useful when only part of AB is wanted.

$$AB = A[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n] = [A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_n] \qquad AB = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{bmatrix} B = \begin{bmatrix} \mathbf{a}_1 B \\ \mathbf{a}_2 B \\ \vdots \\ \mathbf{a}_m B \end{bmatrix}$$

> jth column of $AB = A \cdot (\, j\text{th column of } B\,)$ and ith row of $AB = (\, i\text{th row of } A\,) \cdot B$

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

THEOREM 1.3.1

$A\mathbf{x}$ is a linear combination of the columns of A

If A is $m \times n$ and $\mathbf{x}$ is an $n \times 1$ column vector, then $A\mathbf{x}$ is the linear combination of the COLUMNS of A whose coefficients are the entries of $\mathbf{x}$.

Example 8

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - 1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

- This is the single most reused idea in the course.

EXAMPLE 10

Column-row expansion

Partition A into its columns and B into its rows. Then AB is a SUM of products, each a column times a row:

$$AB = \mathbf{c}_1\mathbf{r}_1 + \mathbf{c}_2\mathbf{r}_2 + \cdots + \mathbf{c}_r\mathbf{r}_r$$

$$\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 4 \\ -3 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} 2 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix}\begin{bmatrix} -3 & 5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 4 \\ 4 & 0 & 8 \end{bmatrix} + \begin{bmatrix} -9 & 15 & 3 \\ 3 & -5 & -1 \end{bmatrix} = \begin{bmatrix} -7 & 15 & 7 \\ 7 & -5 & 7 \end{bmatrix}$$

- Each term is a full $m \times n$ matrix. This expansion is used mainly to prove things, not to compute.

The matrix form of a linear system

Two matrices are equal exactly when their entries are equal, so the m equations of a linear system collapse into ONE matrix equation:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

> $A\mathbf{x} = \mathbf{b}$, where A is the COEFFICIENT matrix, $\mathbf{x}$ the matrix of UNKNOWNS, and $\mathbf{b}$ the matrix of CONSTANTS.

- By Theorem 1.3.1, asking whether $A\mathbf{x} = \mathbf{b}$ is consistent is asking whether $\mathbf{b}$ is a linear combination of the columns of A.

Four equivalent ways to write one system

All four say the same thing about $2x_1 + 3x_2 = 7$, $x_1 - x_2 = 5$. Each makes a different question easy.

1. As equations

$$\begin{array}{ccccc} 2x_1 & + & 3x_2 & = & 7 \\ x_1 & - & x_2 & = & 5 \end{array}$$

2. As an augmented matrix

$$\begin{bmatrix} 2 & 3 & 7 \\ 1 & -1 & 5 \end{bmatrix}$$

3. As a vector equation

$$x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

4. As a matrix equation

$$\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

- Form 2 is for SOLVING, form 3 is for asking about SPAN, form 4 is for asking about INVERTIBILITY. Be able to move between them in both directions.

DEFINITION

The transpose

The TRANSPOSE $\mathbf{A^T}$ of an $m \times n$ matrix A is the $n \times m$ matrix whose columns are the rows of A: $\left(\mathbf{A^T}\right)_{\mathbf{ij}} = (\mathbf{A})_{\mathbf{ji}}$.

$$\underset{B}{\begin{bmatrix} 2 & -1 & 5 \\ 0 & 3 & 4 \end{bmatrix}} \longrightarrow \underset{B^T}{\begin{bmatrix} 2 & 0 \\ -1 & 3 \\ 5 & 4 \end{bmatrix}}$$

- For a square matrix, transposing simply swaps entries in positions mirrored about the main diagonal.

$$\underset{A}{\begin{bmatrix} 1 & 2 & 4 \\ 3 & 7 & 0 \\ 5 & 8 & 6 \end{bmatrix}} \xrightarrow{\text{reflect}} \underset{A^T}{\begin{bmatrix} 1 & 3 & 5 \\ 2 & 7 & 8 \\ 4 & 0 & 6 \end{bmatrix}}$$

DEFINITION

The trace

If A is a SQUARE matrix, the TRACE of A is the sum of the entries on its main diagonal:

$$\operatorname{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}.$$

The trace of a non-square matrix is undefined.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 3 & 5 \\ 5 & -3 & 2 \\ 4 & 0 & 2 \end{bmatrix} \qquad D = [4]$$

$\operatorname{tr}(A) = a + d$ $\quad$ $\operatorname{tr}(B)$ is UNDEFINED B is not square $\quad$ $\operatorname{tr}(C) = 1 + (-3) + 2 = 0$ $\quad$ $\operatorname{tr}(D) = 4$

SECTION 1.4

Inverses; Algebraic Properties of Matrices

GOAL — Find out how far ordinary algebra carries over to matrices, and where it breaks down.

1. The properties that DO hold
2. Three that do NOT: commutativity, cancellation, zero products
3. Zero and identity matrices
4. Invertible and singular matrices
5. The 2×2 inverse formula and $\mathbf{x} = A^{-1}\mathbf{b}$
6. Inverses of products, powers and transposes

THEOREM 1.4.1

Properties of matrix arithmetic

Assuming the sizes are such that the operations can be performed, all of the following hold. Here a and b are scalars.

- $A + B = B + A$
- $A + (B + C) = (A + B) + C$
- $A(BC) = (AB)C$
- $A(B + C) = AB + AC$
- $(B + C)A = BA + CA$
- $A(B - C) = AB - AC$
- $(B - C)A = BA - CA$
- $a(B + C) = aB + aC$
- $a(B - C) = aB - aC$
- $(a + b)C = aC + bC$
- $(a - b)C = aC - bC$
- $a(bC) = (ab)C$
- $a(BC) = (aB)C = B(aC)$

- Because addition and multiplication are associative, we may write $A + B + C$ and ABC with no brackets at all.

WARNING 1

Matrix multiplication is not commutative

In ordinary arithmetic $ab = ba$ always. For matrices AB and BA can differ for THREE separate reasons.

- AB may be defined while BA is not — for instance A is 2×3 and B is 3×4.
- Both may be defined but have DIFFERENT SIZES — A is 2×3 and B is 3×2, so AB is 2×2 and BA is 3×3.
- Both may be defined, the same size, and STILL differ:

$$\begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 11 & 4 \end{bmatrix} \neq \begin{bmatrix} 3 & 6 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}$$

- So AB and BA must be kept apart at all times. Never reorder factors.

Cancellation fails, and zero products happen

You may not cancel a common factor

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix} \Rightarrow AB = AC = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$$

- Here $A \neq 0$ and $AB = AC$, yet $B \neq C$. Cancelling A would be wrong.

A product of nonzero matrices can be zero

$$\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- Neither factor is the zero matrix. So $AB = 0$ does NOT allow you to conclude $A = 0$ or $B = 0$.

Zero matrices and identity matrices

The zero matrix behaves like the number 0

$$A + 0 = 0 + A = A, \qquad A - A = 0, \qquad 0 - A = -A, \qquad A0 = 0, \qquad 0A = 0$$

The identity matrix behaves like the number 1

- I_n is the $n \times n$ matrix with 1's on the main diagonal and 0's elsewhere, and for every $m \times n$ matrix A,

$$AI_n = A \qquad \text{and} \qquad I_mA = A$$

> THEOREM 1.4.3. If R is the reduced row echelon form of an $n \times n$ matrix A, then either R has a row of zeros or $R = I_n$.

- So for a square matrix there are only two possible outcomes of Gauss-Jordan.

DEFINITION

Invertible and singular matrices

Let A be SQUARE. If there is a matrix B of the same size with $AB = BA = I$, then A is INVERTIBLE (or nonsingular) and B is an inverse of A. If no such B exists, A is SINGULAR.

- The condition is symmetric in A and B, so if B is an inverse of A then A is an inverse of B — they are inverses of one another.

Example 5

$$A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \qquad \text{and} \qquad BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

- Both products must be checked in principle. Only square matrices can be invertible.

EXAMPLE 6

A whole class of singular matrices

A square matrix with a row of zeros, or a column of zeros, is SINGULAR. No computation is needed to see it.

$$A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$$

Write $A = [\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{0}]$. For ANY 3×3 matrix B, the column method gives

$$BA = B[\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{0}] = [B\mathbf{c}_1 \quad B\mathbf{c}_2 \quad \mathbf{0}] = \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 0 \end{bmatrix}$$

- That third column of zeros survives whatever B is, so BA can never equal I. Hence A is singular.

THEOREM 1.4.4

An invertible matrix has exactly one inverse

> If B and C are both inverses of A, then $B = C$. We may therefore speak of THE inverse, written A^{-1}, with $AA^{-1} = I$ and $A^{-1}A = I$.

Proof

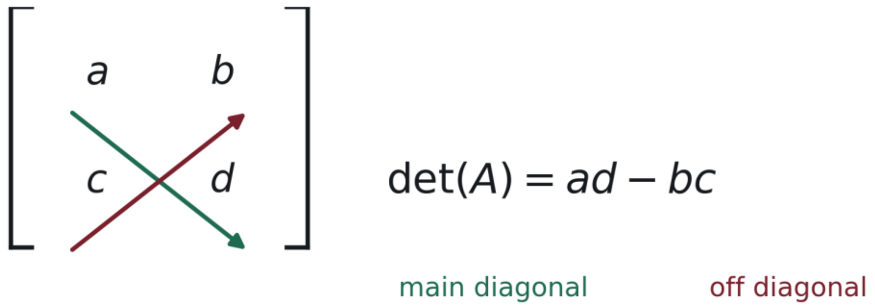
$$B = BI = B(AC) = (BA)C = IC = C$$

- Every step is one of the properties from Theorem 1.4.1 — the identity, then associativity. Worth memorising as a model of how matrix proofs go.
- WARNING. The symbol A^{-1} must NOT be read as $1/A$. Division by a matrix is never defined in this course.

THEOREM 1.4.5

Inverting a 2×2 matrix

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$, and then $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.



A is invertible exactly when $\det(A) \neq 0$

- Swap the diagonal entries, negate the off-diagonal entries, divide by the determinant. A general method for any size comes in §1.5.

EXAMPLE 7

Checking for and Computing a 2×2 inverse

(a) $A = \begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}$

$$\det(A) = (4)(3) - (2)(6) = 0$$

- A is SINGULAR — it has no inverse.

(b) $A = \begin{bmatrix} -3 & -1 \\ 4 & 2 \end{bmatrix}$

$$\det(A) = (-3)(2) - (-1)(4) = -2$$

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} 2 & 1 \\ -4 & -3 \end{bmatrix} = \begin{bmatrix} -1 & -\frac{1}{2} \\ 2 & \frac{3}{2} \end{bmatrix}$$

Always check: $\begin{bmatrix} -3 & -1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} -1 & -\frac{1}{2} \\ 2 & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Solving $Ax = \mathbf{b}$ with the inverse

> If A is invertible, then $A\mathbf{x} = \mathbf{b}$ has the UNIQUE solution $\mathbf{x} = A^{-1}\mathbf{b}$.

- It is a solution: $A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = I\mathbf{b} = \mathbf{b}$.
- It is the only one: if $A\mathbf{u} = \mathbf{b}$, multiply on the LEFT by A^{-1} to get $\mathbf{u} = A^{-1}\mathbf{b}$.

Example — reusing the inverse from Example 7(b)

$$\begin{array}{rcrcr} -3x & - & y & = & 1 \\ 4x & + & 2y & = & 0 \end{array} \quad \Leftrightarrow \quad \begin{bmatrix} -3 & -1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & -\frac{1}{2} \\ 2 & \frac{3}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

- Multiply on the LEFT and keep $\mathbf{b}$ on the right — $\mathbf{b}A^{-1}$ is not even defined here.

THEOREM 1.4.6

The inverse of a product reverses the order

> If A and B are invertible of the same size, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof

$$(AB)\left(B^{-1}A^{-1}\right) = A\left(BB^{-1}\right)A^{-1} = AIA^{-1} = AA^{-1} = I$$

- The same computation in the other order gives $\left(B^{-1}A^{-1}\right)(AB) = I$, so $B^{-1}A^{-1}$ really is the inverse.
- The rule extends to any number of factors: $(A_1A_2\cdots A_k)^{-1} = A_k^{-1}\cdots A_2^{-1}A_1^{-1}$.
- The ORDER REVERSAL is essential and is a favourite exam trap. $(AB)^{-1}$ is almost never $A^{-1}B^{-1}$.

Powers of a matrix

For a square matrix A and a positive integer n,

$$A^0 = I, \qquad A^n = \underbrace{AA\cdots A}_{n \text{ factors}}, \qquad A^{-n} = \left(A^{-1}\right)^n \quad \text{(If } A \text{ is invertible)}$$

- The usual exponent laws hold: $A^r A^s = A^{r+s}$ and $(A^r)^s = A^{rs}$.

If A is invertible and $n \geq 0$: A^{-1} is invertible with $\left(A^{-1}\right)^{-1} = A$; A^n is invertible with $(A^n)^{-1} = \left(A^{-1}\right)^n$; and for a nonzero scalar k, kA is invertible with $(kA)^{-1} = \frac{1}{k}A^{-1}$.

Squaring a sum, and matrix polynomials

Because AB and BA need not agree, the familiar expansion stops one step early:

$$(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2$$

- **Only when** $AB = BA$ may this be written $A^2 + 2AB + B^2$.

Matrix polynomials

- For $p(x) = a_0 + a_1 x + \cdots + a_m x^m$ define $p(A) = a_0 I + a_1 A + \cdots + a_m A^m$
- Note the constant term becomes $a_0 I$, not the scalar a_0.

$$p(x) = x^2 - 2x - 3, \qquad A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} \Rightarrow p(A) = A^2 - 2A - 3I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Transposes, and the inverse of a transpose

Properties of the transpose

- $(A^T)^T = A$
- $(A+B)^T = A^T + B^T$
- $(A-B)^T = A^T - B^T$
- $(kA)^T = kA^T$
- $(AB)^T = B^T A^T$

Inverse of a transpose

- If A is invertible then so is A^T, and

$$(A^T)^{-1} = (A^{-1})^T$$

- Proof: transpose $AA^{-1} = I$ to get $(A^{-1})^T A^T = I$.

> Both $(AB)^T = B^T A^T$ and $(AB)^{-1} = B^{-1}A^{-1}$ REVERSE the order. Transpose and inverse, taken together, do not.

Unit 1B in one slide

- A matrix is an array with a size; $(A)_{ij} = a_{ij}$, and equality needs the same size AND the same entries.
- Addition, subtraction and scalar multiplication are entrywise; multiplication is not.
- AB exists when the INSIDE sizes match, and the OUTSIDE sizes give its shape; $(AB)_{ij}$ is row i of A against column j of B.
- $A\mathbf{x}$ is a linear combination of the columns of A, which turns any system into $A\mathbf{x} = \mathbf{b}$.
- Most rules of algebra survive, but commutativity, cancellation and the zero-product rule do not.
- A is invertible when some B has $AB = BA = I$; the inverse is unique, and for 2×2 it exists exactly when $ad - bc \neq 0$.
- Inverse and transpose both REVERSE the order of a product.

NEXT — Section 1.5: elementary matrices and finding A^{-1} by row reduction.