1.3 Matrices and Matrix Operations

- A **matrix** is a rectangular array of numbers. The numbers in the array are called the entries in the matrix.
- If a matrix A has m **rows** (horizontal lines) n **columns** (vertical lines), then we say A is a **size** $m \times n$ or $A \in M_{m\times n}$.

Example: Types of Matrices

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}, \quad [2 \quad 1 \quad 0 \quad -3], \quad \begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad [4] \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3×2

row matrix
1×4

3×3
square matrix

column matrix
2×1

1×1
row matrix
column matrix
Square matrix

zero matrix
2×4

Matrix General Form:

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \in M_{m\times n}$$

number of rows: m

number of columns: n

size: $m \times n$

(*i*, *j*)-th entry (or element): $(A)_{ij} = a_{ij}$

Square matrix: $m = n$

Example: If $A = \begin{bmatrix} 2 & 4 & -1 \\ 1 & 3 & 0 \end{bmatrix}$, then $(A)_{22} = 3$ and $(A)_{13} = -1$.

Types of Square Matrices: A matrix $A \in M_{n\times n}$ is called

- **Upper triangular** if all entries below its diagonal are 0:

  $(A_{ij} = 0$ whenever $i > j)$

  $$[3], \quad \begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 5 & 0 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & -1 \end{bmatrix}$$

- **Lower triangular** if all its entries above the diagonal are 0:

  $(A_{ij} = 0$ whenever $i < j)$

  $$[3], \quad \begin{bmatrix} 0 & 0 \\ 6 & 3 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 7 & 0 & -1 \end{bmatrix}$$

- **Diagonal** if its both upper and lower triangular:

  $(A_{ij} = 0$ whenever $i \neq j)$

  $$[3], \quad \begin{bmatrix} 11 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- **Identity** if it's a diagonal matrix with all diagonal entries are 1

  $$I_1 = [1], \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \ldots$$

Equal Matrices:

Two matrices are equal if they have the same size $m \times n$ and entries corresponding to the same position are equal, i.e., $A_{ij} = B_{ij}$ for $1 \le i \le m, 1 \le j \le n$.

Example: Equality of matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If $A = B$, then $a = 1$, $b = 2$, $c = 3$, and $d = 4$

Addition and Scalar Multiplication: Let $A, B \in M_{m\times n}$

We define $A + B$ to be the matrix whose ij-th entry is $A_{ij} + B_{ij}$. In other words,

$$(A+B)_{ij} = (A)_{ij} + (B)_{ij}$$

That is, to add two matrices, we add their corresponding entries.

For any $t \in \mathbb{R}$, we define scalar multiplication of A by t to be the matrix whose ij-th entry is $(tA)_{ij}$. In other words,

$$(tA)_{ij} = t(A)_{ij}$$

That is, we multiply a matrix by a scalar by multiplying each entry of the matrix by the scalar.

We define $A - B$ to be the matrix whose ij-th entry is $A_{ij} - B_{ij}$. In other words,

$$(A-B)_{ij} = (A)_{ij} - (B)_{ij}$$

Note that $A - B = A + (-1)B$.

Example: Addition and Scalar Multiplication

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ -1 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 5 \\ -2 & 4 \\ 7 & -3 \end{bmatrix}$.

Then,

$$A + B = \begin{bmatrix} 1+0 & 3+5 \\ 2+(-2) & 6+4 \\ (-1)+7 & (-5)+(-3) \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 0 & 10 \\ 6 & -8 \end{bmatrix}$$

and

$$\sqrt{2}A = \begin{bmatrix} \sqrt{2} & 3\sqrt{2} \\ 2\sqrt{2} & 6\sqrt{2} \\ -\sqrt{2} & -5\sqrt{2} \end{bmatrix}$$

Example: Linear Combination of Matrices

Let $A_1 = \begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, and $A_3 = \begin{bmatrix} 3 & 1 & 1 \\ -2 & -1 & 7 \end{bmatrix}$.

Then the linear combination

$$\begin{aligned} 2A_1 - A_2 + A_3 &= 2\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & 0 \end{bmatrix} + (-1)\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 1 \\ -2 & -1 & 7 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 4 & 2 \\ 6 & -2 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 1 \\ -2 & -1 & 7 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 5 & 2 \\ 4 & -4 & 6 \end{bmatrix} \end{aligned}$$

Theorem: Properties of Addition and scalar Multiplication

For any $A, B, C \in M_{m\times n}$ and any $s, t \in \mathbb{R}$:

1. $A + B \in M_{m\times n}$, **closed** under addition.
2. $A + B = B + A$, addition is **commutative**
3. $(A + B) + C = A + (B + C)$, addition is **associative**
4. There exists a zero matrix O_{mn}, such that $A + O_{mn} = A$, additive **identity**.
5. There exists a matrix $-A \in M_{m\times n}$ such that $A + (-A) = O_{mn}$,additive **inverse**.

6. $sA \in M_{m\times n}$, **closed** under scalar multiplication.
7. $s(tA) = (st)A$, scalar multiplication is **associative**.
8. $(s + t)A = sA + tA$ **matrix distribution**.
9. $s(A + B) = sA + sB$, **scalar distribution**.
10. $1A = A$, **scalar multiplicative identity**.

Proof of property 3:

for any $1 \leq i \leq m$ and $1 \leq j \leq n$,

$$
\begin{aligned}
&((A+B)+C)_{ij} \\
&= (A+B)_{ij} + (C)_{ij} && \text{by definition of addition} \\
&= ((A)_{ij} + (B)_{ij}) + (C)_{ij} && \text{by definition of addition} \\
&= (A)_{ij} + ((B)_{ij} + (C)_{ij}) && \text{by associativity of addition of real numbers} \\
&= (A)_{ij} + (B+C)_{ij} && \text{by definition of addition} \\
&= (A+(B+C))_{ij} && \text{by definition of addition}
\end{aligned}
$$

And since $((A+B)+C)_{ij} = (A+(B+C))_{ij}$ for all applicable i and j, the definition of equality tells us that $(A+B)+C = A+(B+C)$. $\square$

Matrix Multiplication:

Multiplying a **row** matrix by a **column** matrix of the **same length**:

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_m \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{bmatrix} = [y_1 z_1 + y_2 z_2 + \cdots + y_m z_m]$$

Example: Multiplying a row matrix by a column matrix

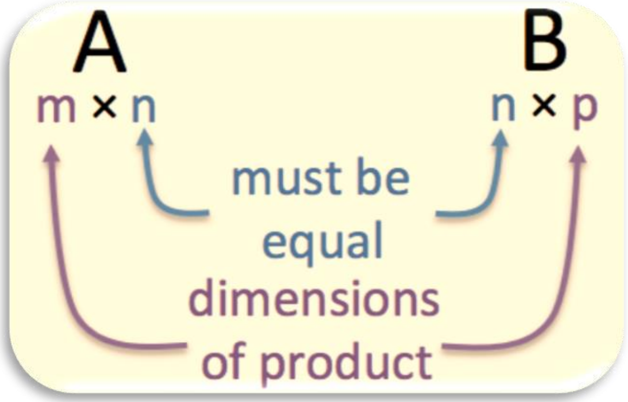
$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = [1 \cdot (-1) + 2 \cdot 0 + 3 \cdot 1] = [2]$$

Multiplying matrices in general: Let A be $m \times n$ and B be $n \times p$ matrices.

AB is the $m \times p$ matrix found by multiplying every row of A by every column of B. If $row_i(A)$ is the i[th] row of A and $col_j(B)$ is the j[th] column of B, then

$$(AB)_{ij} = row_i(A) \cdot col_j(B) = A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{in}B_{nj}$$

$$AB = \begin{bmatrix} row_1(A)\cdot col_1(B) & row_1(A)\cdot col_2(B) & \cdots & row_1(A)\cdot col_p(B) \\ row_2(A)\cdot col_1(B) & row_2(A)\cdot col_2(B) & \cdots & row_2(A)\cdot col_p(B) \\ \vdots & \vdots & \ddots & \vdots \\ row_m(A)\cdot col_1(B) & row_m(A)\cdot col_2(B) & \cdots & row_m(A)\cdot col_p(B) \end{bmatrix}$$



Example: Multiplying matrices

$$\begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} \square & \square & 2(6)+3(3) \\ \square & \square & \square \end{bmatrix} = \begin{bmatrix} \square & \square & 21 \\ \square & \square & \square \end{bmatrix}$$

Note that:

$$AB = \begin{bmatrix} row_1(A)\cdot B \\ row_2(A)\cdot B \\ \vdots \\ row_m(A)\cdot B \end{bmatrix} = \begin{bmatrix} A\cdot col_1(B) & A\cdot col_2(B) & \cdots & A\cdot col_p(B) \end{bmatrix}$$

- $row_i(AB) = row_i(A)\cdot B$

Example:

$$AB = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

- $col_j(AB) = A\cdot col_j\,(B)$

Example:

$$AB = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

Matrix Product as a linear Combination: Let $A \in M_{m\times n}$

$$A\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 col_1(A) + x_2 col_2(A) + \cdots + x_n col_n(A)$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Then

$$A\mathbf{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = x_1\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Matrix Form of a Linear System

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

⇓

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \qquad \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

⇓

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

⇓

$$Ax = b$$

A: the coefficient matrix
x: the unknown matrix
b: the constant matrix

Notes: We now have four **equivalent** ways of expressing linear systems.

1. A system of equations:

$$\begin{aligned} 2x_1 + 3x_2 &= 7 \\ x_1 - x_2 &= 5 \end{aligned}$$

2. An augmented matrix:

$$\left[\begin{array}{cc|c} 2 & 3 & 7 \\ 1 & -1 & 5 \end{array}\right]$$

3. A vector equation:

$$x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

4. As a matrix equation:

$$\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

Each representation gives us a different way to think about linear systems.

Matrix Product as Column-Row Expansion: If A is $m \times n$ and B is $n \times p$ matrix.

$$AB = col_1(A)\cdot row_1(B) + col_2(A)\cdot row_2(B) + \cdots + col_n(A)\cdot row_n(B)$$

Example: Column-Row Expansion

Find the column-row expansion of the product $AB = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & 3 \end{bmatrix}$

Sol.

$$\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}\begin{bmatrix} 1 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} 0 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 6 & 12 \\ 5 & 10 & 20 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 3 \\ 0 & -2 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 15 \\ 5 & 8 & 26 \end{bmatrix}$$

Theorem: Properties of Matrix Multiplication

Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

a. $A(BC) = (AB)C$ (associative law of multiplication)

b. $A(B + C) = AB + AC$ (left distributive law)

c. $(B + C)A = BA + CA$ (right distributive law)

d. $r(AB) = (rA)B = A(rB)$
for any scalar r

e. $I_m A = A = A I_n$ (identity for matrix multiplication)

WORNINGS

1. Matrix Multiplication is **not commutative**, i.e. in general, $AB \neq BA$:

- AB maybe defined and BA may not, e.g., A is 2×3 and B is 3×4.
- AB and BA may have different sizes, e.g., A is 2×3 and B is 2×3.
- $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & * \\ * & * \end{bmatrix} \neq \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & * \\ * & * \end{bmatrix}$

2. The product of nonzero matrices can be a zero matrix, i.e., $AB = 0 \nRightarrow (A = 0 \; or \; B = 0)$:

$$\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

3. The cancelation law does not hold for matrix multiplication, i.e., $AB = AC \nRightarrow B = C$:

$$\begin{bmatrix} -2 & 3 \\ 4 & -6 \end{bmatrix} \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 4 & -6 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 7 \\ 2 & -14 \end{bmatrix}$$

The Transpose of a Matrix A:

A^T is the matrix whose columns are the rows of A. That is $(A^T)_{ij} = A_{ji}$.

Example: Some transposes

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} -5 & 2 \\ 1 & -3 \\ 0 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -3 & 5 & -2 & 7 \end{bmatrix}$

Then $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, $B^T = \begin{bmatrix} -5 & 1 & 0 \\ 2 & -3 & 4 \end{bmatrix}$, $C^T = \begin{bmatrix} 1 & -3 \\ 1 & 5 \\ 1 & -2 \\ 1 & 7 \end{bmatrix}$

The Trace of a Square Matrix A of size n:

$tr(A)$ is the sum of **main diagonal** entries of A. That is $tr(A) = A_{11} + A_{22} + \cdots + A_{nn}$.

Example: Finding the trace

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 3 & 5 \\ 5 & -3 & 2 \\ 4 & 0 & 2 \end{bmatrix}$, $D = [4]$.

Then $\text{tr}(A) = a + d$, $\text{tr}(B)$ is not defined, $\text{tr}(C) = 0$, $\text{tr}(D) = 4$.

Theorem: Properties of Transposes

Let A and B denote matrices whose sizes are appropriate for the following sums and products.

a. $(A^T)^T = A$
b. $(A + B)^T = A^T + B^T$
c. For any scalar r, $(rA)^T = rA^T$
d. $(AB)^T = B^T A^T$

Theorem: Properties of Trace

Let A and B be square matrices of the same size.
a. $tr(A + B) = tr(A) + tr(B)$
b. $tr(sA) = str(A)$
c. $tr(A^T) = tr(A)$

1.4 Inverses, More Algebraic Properties of Matrices

Theorem: If $A \in M_{n\times n}$, then either $rref(A)$ has a row of zeros or $rref(A) = I_n$.

Proof. Either the last row in $rref(A)$ is zero otherwise it contains no zero rows, and consequently each of the n rows has a leading entry of 1. This implies that each of the n columns contains a leading 1. Since these leading 1's occur progressively farther to the right as we move down, each of these 1's must occur on the main diagonal.

Definition: If A is a square matrix for which there is a matrix of the same size, say B, such that $AB = BA = I$ then A is called invertible and B is called its inverse. If we can't find such a matrix B, then A is called a singular matrix.

Example: An invertible Matrix

$A = \begin{bmatrix} -2 & -1 \\ 5 & 3 \end{bmatrix}$ is invertible since $\mathrm{B} = \begin{bmatrix} -3 & -1 \\ 5 & 2 \end{bmatrix}$ satisfies:

$$AB = \begin{bmatrix} -2 & -1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} -3 & -1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} -3 & -1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Example: A singular Matrix

$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{bmatrix}$ is singular, because if B is any 3×3 matrix we must have

$$AB = \begin{bmatrix} * & * & * \\ 0 & 0 & 0 \\ * & * & * \end{bmatrix} \neq I$$

Note : Any matrix with a zero row (or column) is singular.

Theorem Inverse is Unique

If both B and C are inverses of a matrix A, then $B = C$. The inverse is denoted A^{-1}.

Proof: Observe that $B = BI = B(AC) = (BA)C = IC = C$.□

Theorem Invertibility for 2×2 Matrices

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible $iff\ ad - bc \neq 0$, in which case $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Example: Calculating the Inverse of a 2×2 Matrix

Find the inverse if it exists: a) $A = \begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}$. b) $A = \begin{bmatrix} -3 & -1 \\ 4 & 2 \end{bmatrix}$

Sol: a) Since $(4)(3) - (2)(6) = 0$, A is singular.

Sol: b) Since $(-3)(2) - (-1)(4) = -2 \neq 0$, A is invertible and

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} 2 & 1 \\ -4 & -3 \end{bmatrix} = \begin{bmatrix} -1 & \frac{-1}{2} \\ 2 & \frac{3}{2} \end{bmatrix}.$$

Theorem Solving Linear Systems Using Matrix Inverse

If $A \in M_{n\times n}$ invertible. The equation $Ax = b$ has the unique solution $x = A^{-1}b$.

Proof. Substituting $x = A^{-1}b$ in the equation we get $Ax = A(A^{-1}b) = (AA^{-1})b = Ib = b$. This shows $A^{-1}b$ is a solution. To show it is unique assume u is any solution, i.e., $Au = b$. Then multiplying both sides of this equation by A^{-1}we have $A^{-1}Au = A^{-1}b \Rightarrow Iu = A^{-1}b \Rightarrow u = A^{-1}b$.□

Example Solving Linear Systems Using Matrix Inverse

Use matrix inverse to solve the linear system: $\begin{array}{c} -3x - y = 1 \\ 4x + 2y = 0 \end{array}$.

Sol This system is given by $\begin{bmatrix} -3 & -1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We know

$\begin{bmatrix} -3 & -1 \\ 4 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & \frac{-1}{2} \\ -2 & \frac{3}{2} \end{bmatrix}$ so the solution is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & \frac{-1}{2} \\ -2 & \frac{3}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$.

Power of a Matrix

- **Zero Power of a Matrix** $A \in M_{n\times n}$ is define by $A^0 = I$.
- **A Positive Power of a Matrix** $A \in M_{n\times n}$, is define by $A^m = AA \cdots A$ (m factors).
- For $k, l \in \{0,1,2\ \ldots\}$, We also have $A^k A^l = A^{k+l}$ and $(A^k)^l = A^{kl}$.
- **A Negative Power of an Invertible Matrix** $A \in M_{n\times n}$, is defined by

$$A^{-m} = A^{-1}A^{-1} \cdots A^{-1} \text{ (m factors).}$$

Example Squaring a Matrix Sum

- If $A, B \in M_{n\times n}$, then $(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2$
- If further $AB = BA$, then $(A+B)^2 = A^2 + 2AB + B^2$.

Matrix Polynomials If $A \in M_{n\times n}$ and $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$, then define:

$$p(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_m A^m$$

Example Matrix Polynomial

Compute $p(A)$ where $p(x) = x^2 + 3x + 2$ and $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$.

Note that if $f(x) = p(x)q(x)$ and $A \in M_{n\times n}$, then $f(A) = p(A)q(A) = q(A)p(A)$.

Theorem: Matrix Inverse Relationship with other operations

$A, B \in M_{n\times n}$ invertible. Then

1. AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
2. A^{-1} is invertible and $(A^{-1})^{-1} = A$.
3. For $m \in \{0,1,2\ \dots\}$, A^m is invertible and $(A^m)^{-1} = A^{-m} = (A^{-1})^m$.
4. If $s \in \mathbb{R}$ nonzero scalar, then sA is invertible and $(sA)^{-1} = \frac{1}{s}A^{-1}$.
5. A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

Proof: