# The scaling properties and the multiple derivative of Legendre polynomials 

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#### Abstract

In this paper, we study the scaling properties of Legendre polynomials $P_{n}(x)$. We show that $P_{n}(\lambda x)$, where $\lambda$ is a constant, can be expanded as a sum of either Legendre polynomials $P_{n}(x)$ or their multiple derivatives $d^{k} P_{n}(x) / d x^{k}$, and we derive a general expression for the expansion coefficients. In addition, we demonstrate that the multiple derivative $d^{k} P_{n}(x) / d x^{k}$ can also be expressed as a sum of Legendre polynomials and we obtain a recurrence relation for the coefficients.


## KEYWORDS

Legendre Polynomials, scaling property, multiple derivative, expansion.

## 1. Introduction

The central importance of Legendre polynomials in many fields of pure and applied sciences is undoubtedly well-established. Consequently, their properties have been extensively investigated over several decades [1.|2], and still remain a matter for numerous studies [3-8]. To our knowledge, though, the scaling properties of Legendre polynomials, hereafter denoted as $P_{n}(x)$, have not been reported in the literature so far. The purpose of this work is to derive explicit expressions for $P_{n}(\lambda x)$, where $\lambda$ is a constant. Such expressions appear to be very useful for deriving an analytical representation for the projection of spherical harmonics onto a plan. In this paper, we show that $P_{n}(\lambda x)$ can be expanded either as a sum of Legendre polynomials $P_{n}(x)$ or their multiple derivatives:

$$
\begin{equation*}
P_{n}(\lambda x)=\sum_{k=0}^{\lfloor n / 2\rfloor} a_{\lambda, n, k} \frac{d^{k}}{d x^{k}} P_{n-k}(x)=\sum_{k=0}^{\lfloor n / 2\rfloor} b_{\lambda, n, k} P_{n-2 k}(x), \tag{1}
\end{equation*}
$$

and we derive an expression for the expansion coefficients $a_{\lambda, n, k}$ and $b_{\lambda, n, k}$. In addition, we demonstrate that the multiple $\left(k^{t h}\right)$ derivative of Legendre polynomials of degree $n$ can also be expressed as a sum of Legendre polynomials:

$$
\begin{equation*}
\frac{d^{k}}{d x^{k}} P_{n}(x)=\sum_{i=0}^{\lfloor(n-k) / 2\rfloor} \alpha_{n-k-2 i} P_{n-k-2 i}(x), \tag{2}
\end{equation*}
$$

and we derive a recurrence relation for the coefficients $\alpha_{n-k-2 i}$.

## 2. Proof

### 2.1. Scaling properties

Using Rodrigues' formula, Legendre polynomials $P_{n}(\lambda x)$ of degree $n$ can be written as [1]:

$$
\begin{equation*}
P_{n}(\lambda x)=\frac{1}{2^{n} n!} \frac{d^{n}}{d(\lambda x)^{n}}\left[\left(\lambda^{2} x^{2}-1\right)^{n}\right] \tag{3}
\end{equation*}
$$

By recognizing that $d(\lambda x)^{n}=\lambda^{n} d x^{n}$ and $\lambda^{2} x^{2}-1=\lambda^{2}\left(x^{2}-1\right)+\lambda^{2}-1$, equation (3) can be rewritten as:

$$
\begin{equation*}
P_{n}(\lambda x)=\frac{1}{2^{n} n!} \lambda^{n} \frac{d^{n}}{d x^{n}}\left[\left(\left(x^{2}-1\right)+\frac{\lambda^{2}-1}{\lambda^{2}}\right)^{n}\right] \tag{4}
\end{equation*}
$$

In the previous equation, the polynomial of degree $n$ being derived can be expanded as a sum of $n+1$ terms:

$$
\begin{equation*}
\left(\left(x^{2}-1\right)+\frac{\lambda^{2}-1}{\lambda^{2}}\right)^{n}=\sum_{k=0}^{n} \alpha_{k}\left(x^{2}-1\right)^{k} \tag{5}
\end{equation*}
$$

whose expansion coefficients, $\alpha_{k}$, can be related to the product of its unique root $\left(\lambda^{2}-1\right) / \lambda^{2}$ using Vieta's formula [9]. It can be shown that:

$$
\begin{equation*}
\alpha_{k}=\binom{n}{k}\left(\frac{\lambda^{2}-1}{\lambda^{2}}\right)^{n-k} \tag{6}
\end{equation*}
$$

From equations (4), (5), and (6), it follows that:

$$
\begin{equation*}
P_{n}(\lambda x)=\frac{1}{2^{n} n!} \frac{d^{n}}{d x^{n}} \sum_{k=0}^{n}\binom{n}{k} \frac{\left(\lambda^{2}-1\right)^{n-k}}{\lambda^{n-2 k}}\left(x^{2}-1\right)^{k} \tag{7}
\end{equation*}
$$

By interchanging the order of the derivation and the sum in equation (7) and recognizing that the $n^{\text {th }}$ derivative of the polynomial $\left(x^{2}-1\right)^{k}$ vanishes for $k$ lower than $\lceil n / 2\rceil$, we obtain:

$$
\begin{equation*}
P_{n}(\lambda x)=\sum_{k=\lceil n / 2\rceil}^{n} \frac{1}{2^{n} n!}\binom{n}{k} \frac{\left(\lambda^{2}-1\right)^{n-k}}{\lambda^{n-2 k}} \frac{d^{n}}{d x^{n}}\left(x^{2}-1\right)^{k} \tag{8}
\end{equation*}
$$

Recognizing the Rodrigues representation of the Legendre polynomial of degree $k$ in the previous equation and substituting for $P_{k}(x)$, we obtain:

$$
\begin{equation*}
P_{n}(\lambda x)=\sum_{k=\lceil n / 2\rceil}^{n} \frac{1}{2^{n-k}(n-k)!} \frac{\left(\lambda^{2}-1\right)^{n-k}}{\lambda^{n-2 k}} \frac{d^{n-k}}{d x^{n-k}} P_{k}(x) \tag{9}
\end{equation*}
$$

which can be rewritten as:

$$
\begin{equation*}
P_{n}(\lambda x)=\sum_{k=0}^{\lfloor n / 2\rfloor} a_{\lambda, n, k} \frac{d^{k}}{d x^{k}} P_{n-k}(x) \tag{10}
\end{equation*}
$$

with

$$
\begin{equation*}
a_{\lambda, n, k}=\frac{\lambda^{n-2 k}\left(\lambda^{2}-1\right)^{k}}{2^{k}(k)!} \tag{11}
\end{equation*}
$$

Equation (10) shows that $P_{n}(\lambda x)$ can be expressed as a sum of multiple derivatives of Legendre polynomials, which, in turn, can be expanded as a sum of Legendre Polynomials, as proved in the following section. It will be shown that:

$$
\begin{equation*}
\frac{d^{k}}{d x^{k}} P_{n-k}(x)=\sum_{i=0}^{\lfloor(n-2 k) / 2\rfloor} \alpha_{n-2 k-2 i} P_{n-2 k-2 i}(x) \tag{12}
\end{equation*}
$$

with

$$
\begin{align*}
\alpha_{n-2 k-2 i}= & \frac{2^{k+2 i}(n-k-1 / 2)^{\underline{k}}(n-k-i)^{\underline{i}}(n-2 k-1 / 2)^{2 i}}{(2 i)^{\underline{2}}(n-k-1 / 2) \underline{i}} \\
& -\sum_{l=0}^{i-1} \frac{(2(n-2 k-i-l)) \frac{2(i-l)}{(2(i-l)) \frac{2(i-l)}{}} \alpha_{n-2 k-2 l}}{} \tag{13}
\end{align*}
$$

Combining equations (10), (11), and (12), it follows that:

$$
\begin{equation*}
P_{n}(\lambda x)=\sum_{k=0}^{\lfloor n / 2\rfloor} \frac{\lambda^{n-2 k}\left(\lambda^{2}-1\right)^{k}}{2^{k}(k)!} \sum_{i=0}^{\lfloor(n-2 k) / 2\rfloor} \alpha_{n-2 k-2 i} P_{n-2 k-2 i}(x) \tag{14}
\end{equation*}
$$

Finally, after rearranging the terms in the second sum of the previous equation, we obtain the following:

$$
\begin{equation*}
P_{n}(\lambda x)=\sum_{k=0}^{\lfloor n / 2\rfloor} b_{\lambda, n, k} P_{n-2 k}(x) \tag{15}
\end{equation*}
$$

where,

$$
\begin{equation*}
b_{\lambda, n, k}=\sum_{i=0}^{\max \{(k-1), 0\}} \frac{\lambda^{n-2 k+2 i}}{2^{k-i}(k-i)!}\left(\lambda^{2}-1\right)^{k-i} \alpha_{n, k, i} \tag{16}
\end{equation*}
$$

and

$$
\begin{gather*}
\alpha_{n, k, i}=\frac{2^{k+i}(n-k+i-1 / 2) \frac{k-i}{}(n-k)^{\underline{i}}(n-2 k+2 i-1 / 2)^{2 i}}{(2 i)^{\underline{2 i}}(n-k+i-1 / 2)^{\underline{i}}} \\
-\sum_{\substack{l=0 \\
i \neq 0}}^{i-1} \frac{(2(n-2 k+i-l))^{2(i-l)}}{(2(i-l))^{2(i-l)}} \alpha_{n, k-i+l, l} . \tag{17}
\end{gather*}
$$

### 2.2. Multiple derivative of Legendre polynomials

From the recurrence relations [9]:

$$
\begin{align*}
& (n+1) P_{n+1}(x)-(2 n+1) x P_{n}(x)+n P_{n-1}(x)=0, \quad \text { and }  \tag{18}\\
& \left(x^{2}-1\right) \frac{d}{d x} P_{n}(x)=n\left[x P_{n}(x)-P_{n-1}(x)\right]
\end{align*}
$$

it can be shown that:

$$
\begin{equation*}
\frac{d}{d x}\left[P_{n+1}(x)-P_{n-1}(x)\right]=(2 n+1) P_{n}(x) . \tag{19}
\end{equation*}
$$

From the previous equation, it follows that:

$$
\begin{equation*}
\frac{d}{d x} P_{n}(x)=\frac{2 P_{n-1}(x)}{\left\|P_{n-1}\right\|^{2}}+\frac{2 P_{n-3}(x)}{\left\|P_{n-3}\right\|^{2}}+\frac{2 P_{n-5}(x)}{\left\|P_{n-5}\right\|^{2}}+\ldots \tag{20}
\end{equation*}
$$

where $\left\|P_{n}\right\|=\sqrt{2 /(2 n+1)}$. By recursively deriving equation (20), we finally obtain:

$$
\begin{equation*}
\frac{d^{k}}{d x^{k}} P_{n}(x)=\sum_{i=0}^{\lfloor(n-k) / 2\rfloor} \alpha_{n-k-2 i} P_{n-k-2 i}(x) \tag{21}
\end{equation*}
$$

The expansion coefficients $\alpha_{n-k-2 i}$ can be determined by replacing the Legendre polynomials on each side of the previous equation with their corresponding hypergeometric series [1] and matching the coefficients of each term. Using Murphy's formula, Legendre polynomials can be written as:

$$
\begin{equation*}
P_{n}(x)={ }_{2} F_{1}(-n, n+1,1 ;(1-x) / 2)=\sum_{j=0}^{\infty} \frac{(-n)_{j}(n+1)_{j}}{(1)_{j}} \frac{[(1-x) / 2]^{j}}{j!}, \tag{22}
\end{equation*}
$$

where $(a)_{j}$ denotes the rising factorial. By setting $z=(1-x) / 2$, and recognizing that $d z^{k}=(-1 / 2)^{k} d x^{k}$, the $k^{t h}$ derivative of the Legendre polynomial of degree $n$ with
respect to the variable $x$ is given by:

$$
\begin{align*}
\frac{d^{k}}{d x^{k}} P_{n}(x) & =\frac{d^{k}}{d z^{k}} \frac{d z^{k}}{d x^{k}}{ }^{2} F_{1}(-n, n+1,1 ;(1-x) / 2) \\
& =\left(-\frac{1}{2}\right)^{k} \frac{d^{k}}{d z^{2}} F_{1}(-n, n+1,1 ; z) \\
& =\left(-\frac{1}{2}\right)^{k} \frac{(-n)_{k}(n+1)_{k}}{k!}{ }_{2} F_{1}(-n+k, n+1+k, 1+k ; z) \\
& =\binom{n}{k} \frac{(n+1)_{k}}{2^{k}}{ }_{2} F_{1}(-n+k, n+1+k, 1+k ; z) . \tag{23}
\end{align*}
$$

By recognizing that the rising factorial $(-n+k)_{i}$ is equal to zero for $i \geq n-k+1$, it then follows that:

$$
\begin{equation*}
\frac{d^{k}}{d x^{k}} P_{n}(x)=\binom{n}{k} \frac{(n+1)_{k}}{2^{k}} \sum_{j=0}^{n-k} \frac{(-n+k)_{j}(n+1+k)_{j}}{(1+k)_{j}} \frac{z^{j}}{j!} \tag{24}
\end{equation*}
$$

The same reasoning can be applied to the right side of equation (21), leading to:

$$
\begin{align*}
P_{n-k-2 i}(x) & ={ }_{2} F_{1}(-n+k+2 i, n-k-2 i+1,1 ; z) \\
& =\sum_{j=0}^{n-k-2 i} \frac{(-n+k+2 i)_{j}(n-k-2 i+1)_{j}}{(1)_{j}} \frac{z^{j}}{j!} \tag{25}
\end{align*}
$$

By matching the coefficients of each term in equations (24) and (25), we obtain a set of $(n-k)$ coupled equations:

$$
\begin{align*}
A_{n-k} & =\alpha_{n-k} B_{n-k, n-k}  \tag{26}\\
A_{n-k-1} & =\alpha_{n-k} B_{n-k-1, n-k} \\
A_{n-k-2} & =\alpha_{n-k} B_{n-k-2, n-k}+\alpha_{n-k-2} B_{n-k-2, n-k-2} \\
A_{n-k-3} & =\alpha_{n-k} B_{n-k-3, n-k}+\alpha_{n-k-2} B_{n-k-3, n-k-2} \\
A_{n-k-4} & =\alpha_{n-k} B_{n-k-4, n-k}+\alpha_{n-k-2} B_{n-k-4, n-k-2}+\alpha_{n-k-4} B_{n-k-4, n-k-4} \\
A_{n-k-5} & =\alpha_{n-k} B_{n-k-5, n-k}+\alpha_{n-k-2} B_{n-k-5, n-k-2}+\alpha_{n-k-4} B_{n-k-5, n-k-4}
\end{align*}
$$

where, for the sake of clarity, we have set:

$$
\begin{align*}
A_{j} & =\binom{n}{k} \frac{(n+1)_{k}}{2^{k}} \frac{(-n+k)_{j}(n+1+k)_{j}}{(1+k)_{j}}  \tag{27}\\
B_{j, n-k} & =\frac{(-n+k)_{j}(n-k+1)_{j}}{(1)_{j}} \\
B_{j, n-k-2} & =\frac{(-n+k+2)_{j}(n-k-1)_{j}}{(1)_{j}}=B_{j, n-k} \frac{(n-k-j)^{\underline{2}}}{(n-k+j)^{2}} \\
B_{j, n-k-4} & =\frac{(-n+k+4)_{j}(n-k-3)_{j}}{(1)_{j}}=B_{j, n-k} \frac{(n-k-j)^{\underline{\underline{4}}}}{(n-k+j)^{\underline{4}}} \\
\ldots & \\
B_{j, n-k-2 i} & =\frac{(-n+k+2 i)_{j}(n-k-2 i+1)_{j}}{(1)_{j}}=B_{j, n-k} \frac{(n-k-j)^{\underline{2 i}}}{(n-k+j)^{2 i}}
\end{align*}
$$

In the previous equation, the symbol $(x)^{\underline{n}}$ is used to represent the falling factorial. From equations (26) and identities (27), a recurrence relation for the coefficients $\alpha_{n-k-2 i}$ can be derived. It can be shown that:
$\alpha_{n-k-2 i}=\frac{2^{k+2 i}(n-1 / 2)^{\underline{k}}(n-i) \underline{\underline{i}}(n-k-1 / 2)^{\underline{2}}}{(2 i)^{2 i}(n-1 / 2) \underline{\underline{i}}}-\sum_{l=0}^{i-1} \frac{(2(n-k-i-l))^{2(i-l)}}{(2(i-l))^{2(i-l)}} \alpha_{n-k-2 l}$.

## 3. Conclusion

In this work, we have studied the scaling properties of Legendre polynomials $P_{n}(x)$. We have demonstrated that $P_{n}(\lambda x)$, where $\lambda$ is a constant, can be expanded either as a sum of Legendre polynomials $P_{n}(x)$ or their multiple derivatives, and we have obtained an explicit expression for the expansion coefficients. In addition, we have shown that the multiple derivative $d^{k} P_{n}(x) / d x^{k}$ can also be expressed as a sum of Legendre polynomials and we derived a recurrence relation for the coefficients.

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