The scaling properties and the multiple derivative of Legendre polynomials

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ABSTRACT

In this paper, we study the scaling properties of Legendre polynomials $P_n(x)$. We show that $P_n(\lambda x)$, where λ is a constant, can be expanded as a sum of either Legendre polynomials $P_n(x)$ or their multiple derivatives $d^k P_n(x)/dx^k$, and we derive a general expression for the expansion coefficients. In addition, we demonstrate that the multiple derivative $d^k P_n(x)/dx^k$ can also be expressed as a sum of Legendre polynomials and we obtain a recurrence relation for the coefficients.

KEYWORDS

Legendre Polynomials, scaling property, multiple derivative, expansion.

1. Introduction

The central importance of Legendre polynomials in many fields of pure and applied sciences is undoubtedly well-established. Consequently, their properties have been extensively investigated over several decades [1,2], and still remain a matter for numerous studies [3–8]. To our knowledge, though, the scaling properties of Legendre polynomials, hereafter denoted as $P_n(x)$, have not been reported in the literature so far. The purpose of this work is to derive explicit expressions for $P_n(\lambda x)$, where λ is a constant. Such expressions appear to be very useful for deriving an analytical representation for the projection of spherical harmonics onto a plan. In this paper, we show that $P_n(\lambda x)$ can be expanded either as a sum of Legendre polynomials $P_n(x)$ or their multiple derivatives:

$$P_n(\lambda x) = \sum_{k=0}^{\lfloor n/2 \rfloor} a_{\lambda,n,k} \frac{d^k}{dx^k} P_{n-k}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} b_{\lambda,n,k} P_{n-2k}(x), \tag{1}$$

and we derive an expression for the expansion coefficients $a_{\lambda,n,k}$ and $b_{\lambda,n,k}$. In addition, we demonstrate that the multiple (k^{th}) derivative of Legendre polynomials of degree n can also be expressed as a sum of Legendre polynomials:

$$\frac{d^k}{dx^k}P_n(x) = \sum_{i=0}^{\lfloor (n-k)/2 \rfloor} \alpha_{n-k-2i}P_{n-k-2i}(x), \qquad (2)$$

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and we derive a recurrence relation for the coefficients α_{n-k-2i} .

2. Proof

2.1. Scaling properties

Using Rodrigues' formula, Legendre polynomials $P_n(\lambda x)$ of degree *n* can be written as [1]:

$$P_n(\lambda x) = \frac{1}{2^n n!} \frac{d^n}{d(\lambda x)^n} \left[(\lambda^2 x^2 - 1)^n \right].$$
(3)

By recognizing that $d(\lambda x)^n = \lambda^n dx^n$ and $\lambda^2 x^2 - 1 = \lambda^2 (x^2 - 1) + \lambda^2 - 1$, equation (3) can be rewritten as:

$$P_n(\lambda x) = \frac{1}{2^n n!} \lambda^n \frac{d^n}{dx^n} \left[\left((x^2 - 1) + \frac{\lambda^2 - 1}{\lambda^2} \right)^n \right].$$
(4)

In the previous equation, the polynomial of degree n being derived can be expanded as a sum of n + 1 terms:

$$\left((x^2 - 1) + \frac{\lambda^2 - 1}{\lambda^2}\right)^n = \sum_{k=0}^n \alpha_k (x^2 - 1)^k,$$
(5)

whose expansion coefficients, α_k , can be related to the product of its unique root $(\lambda^2 - 1)/\lambda^2$ using Vieta's formula [9]. It can be shown that:

$$\alpha_k = \binom{n}{k} \left(\frac{\lambda^2 - 1}{\lambda^2}\right)^{n-k}.$$
(6)

From equations (4), (5), and (6), it follows that:

$$P_n(\lambda x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \sum_{k=0}^n \binom{n}{k} \frac{(\lambda^2 - 1)^{n-k}}{\lambda^{n-2k}} (x^2 - 1)^k.$$
(7)

By interchanging the order of the derivation and the sum in equation (7) and recognizing that the n^{th} derivative of the polynomial $(x^2 - 1)^k$ vanishes for k lower than $\lceil n/2 \rceil$, we obtain:

$$P_n(\lambda x) = \sum_{k=\lceil n/2 \rceil}^n \frac{1}{2^n n!} \binom{n}{k} \frac{(\lambda^2 - 1)^{n-k}}{\lambda^{n-2k}} \frac{d^n}{dx^n} (x^2 - 1)^k.$$
(8)

Recognizing the Rodrigues representation of the Legendre polynomial of degree k in the previous equation and substituting for $P_k(x)$, we obtain:

$$P_n(\lambda x) = \sum_{k=\lceil n/2 \rceil}^n \frac{1}{2^{n-k}(n-k)!} \frac{(\lambda^2 - 1)^{n-k}}{\lambda^{n-2k}} \frac{d^{n-k}}{dx^{n-k}} P_k(x), \tag{9}$$

which can be rewritten as:

$$P_n(\lambda x) = \sum_{k=0}^{\lfloor n/2 \rfloor} a_{\lambda,n,k} \frac{d^k}{dx^k} P_{n-k}(x), \qquad (10)$$

with

$$a_{\lambda,n,k} = \frac{\lambda^{n-2k} (\lambda^2 - 1)^k}{2^k (k)!}.$$
(11)

Equation (10) shows that $P_n(\lambda x)$ can be expressed as a sum of multiple derivatives of Legendre polynomials, which, in turn, can be expanded as a sum of Legendre Polynomials, as proved in the following section. It will be shown that:

$$\frac{d^k}{dx^k} P_{n-k}(x) = \sum_{i=0}^{\lfloor (n-2k)/2 \rfloor} \alpha_{n-2k-2i} P_{n-2k-2i}(x)$$
(12)

with

$$\alpha_{n-2k-2i} = \frac{2^{k+2i}(n-k-1/2)\underline{k}(n-k-i)\underline{i}(n-2k-1/2)\underline{2i}}{(2i)\underline{2i}(n-k-1/2)\underline{i}} - \sum_{l=0}^{i-1} \frac{(2(n-2k-i-l))\underline{2(i-l)}}{(2(i-l))\underline{2(i-l)}} \alpha_{n-2k-2l}.$$
 (13)

Combining equations (10), (11), and (12), it follows that:

$$P_n(\lambda x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{\lambda^{n-2k} (\lambda^2 - 1)^k}{2^k (k)!} \sum_{i=0}^{\lfloor (n-2k)/2 \rfloor} \alpha_{n-2k-2i} P_{n-2k-2i}(x).$$
(14)

Finally, after rearranging the terms in the second sum of the previous equation, we obtain the following:

$$P_n(\lambda x) = \sum_{k=0}^{\lfloor n/2 \rfloor} b_{\lambda,n,k} P_{n-2k}(x)$$
(15)

where,

$$b_{\lambda,n,k} = \sum_{i=0}^{\max\{(k-1),0\}} \frac{\lambda^{n-2k+2i}}{2^{k-i}(k-i)!} (\lambda^2 - 1)^{k-i} \alpha_{n,k,i},$$
(16)

and

$$\alpha_{n,k,i} = \frac{2^{k+i}(n-k+i-1/2)\frac{k-i}{(n-k)i}(n-2k+2i-1/2)\frac{2i}{(2i)\frac{2i}{2i}(n-k+i-1/2)\frac{i}{2}}}{-\sum_{\substack{l=0\\i\neq 0}}^{i-1} \frac{(2(n-2k+i-l))\frac{2(i-l)}{(2(i-l))}}{(2(i-l))\frac{2(i-l)}{2}} \alpha_{n,k-i+l,l}.$$
(17)

2.2. Multiple derivative of Legendre polynomials

From the recurrence relations [9]:

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0, \quad \text{and} \quad (18)$$
$$(x^2 - 1)\frac{d}{dx}P_n(x) = n\left[xP_n(x) - P_{n-1}(x)\right],$$

it can be shown that:

$$\frac{d}{dx}\left[P_{n+1}(x) - P_{n-1}(x)\right] = (2n+1)P_n(x).$$
(19)

From the previous equation, it follows that:

$$\frac{d}{dx}P_n(x) = \frac{2P_{n-1}(x)}{||P_{n-1}||^2} + \frac{2P_{n-3}(x)}{||P_{n-3}||^2} + \frac{2P_{n-5}(x)}{||P_{n-5}||^2} + \dots,$$
(20)

where $||P_n|| = \sqrt{2/(2n+1)}$. By recursively deriving equation (20), we finally obtain:

$$\frac{d^k}{dx^k} P_n(x) = \sum_{i=0}^{\lfloor (n-k)/2 \rfloor} \alpha_{n-k-2i} P_{n-k-2i}(x)$$
(21)

The expansion coefficients α_{n-k-2i} can be determined by replacing the Legendre polynomials on each side of the previous equation with their corresponding hypergeometric series [1] and matching the coefficients of each term. Using Murphy's formula, Legendre polynomials can be written as:

$$P_n(x) = {}_2F_1(-n, n+1, 1; (1-x)/2) = \sum_{j=0}^{\infty} \frac{(-n)_j (n+1)_j}{(1)_j} \frac{[(1-x)/2]^j}{j!}, \qquad (22)$$

where $(a)_j$ denotes the rising factorial. By setting z = (1-x)/2, and recognizing that $dz^k = (-1/2)^k dx^k$, the k^{th} derivative of the Legendre polynomial of degree n with

respect to the variable x is given by:

$$\frac{d^{k}}{dx^{k}}P_{n}(x) = \frac{d^{k}}{dz^{k}}\frac{dz^{k}}{dx^{k}}{}_{2}F_{1}(-n,n+1,1;(1-x)/2)$$

$$= \left(-\frac{1}{2}\right)^{k}\frac{d^{k}}{dz^{k}}{}_{2}F_{1}(-n,n+1,1;z)$$

$$= \left(-\frac{1}{2}\right)^{k}\frac{(-n)_{k}(n+1)_{k}}{k!}{}_{2}F_{1}(-n+k,n+1+k,1+k;z)$$

$$= \binom{n}{k}\frac{(n+1)_{k}}{2^{k}}{}_{2}F_{1}(-n+k,n+1+k,1+k;z).$$
(23)

By recognizing that the rising factorial $(-n+k)_i$ is equal to zero for $i \ge n-k+1$, it then follows that:

$$\frac{d^k}{dx^k}P_n(x) = \binom{n}{k}\frac{(n+1)_k}{2^k}\sum_{j=0}^{n-k}\frac{(-n+k)_j(n+1+k)_j}{(1+k)_j}\frac{z^j}{j!}.$$
(24)

The same reasoning can be applied to the right side of equation (21), leading to:

$$P_{n-k-2i}(x) = {}_{2}F_{1}(-n+k+2i, n-k-2i+1, 1; z)$$

=
$$\sum_{j=0}^{n-k-2i} \frac{(-n+k+2i)_{j}(n-k-2i+1)_{j}}{(1)_{j}} \frac{z^{j}}{j!}$$
(25)

By matching the coefficients of each term in equations (24) and (25), we obtain a set of (n - k) coupled equations:

$$A_{n-k} = \alpha_{n-k}B_{n-k,n-k}$$
(26)

$$A_{n-k-1} = \alpha_{n-k}B_{n-k-1,n-k}$$
(26)

$$A_{n-k-2} = \alpha_{n-k}B_{n-k-2,n-k} + \alpha_{n-k-2}B_{n-k-2,n-k-2}$$
(26)

$$A_{n-k-3} = \alpha_{n-k}B_{n-k-2,n-k} + \alpha_{n-k-2}B_{n-k-3,n-k-2}$$
(26)

$$A_{n-k-4} = \alpha_{n-k}B_{n-k-3,n-k} + \alpha_{n-k-2}B_{n-k-3,n-k-2}$$
(26)

where, for the sake of clarity, we have set:

$$A_{j} = \binom{n}{k} \frac{(n+1)_{k}}{2^{k}} \frac{(-n+k)_{j}(n+1+k)_{j}}{(1+k)_{j}}$$

$$B_{j,n-k} = \frac{(-n+k)_{j}(n-k+1)_{j}}{(1)_{j}}$$

$$B_{j,n-k-2} = \frac{(-n+k+2)_{j}(n-k-1)_{j}}{(1)_{j}} = B_{j,n-k} \frac{(n-k-j)^{2}}{(n-k+j)^{2}}$$

$$B_{j,n-k-4} = \frac{(-n+k+4)_{j}(n-k-3)_{j}}{(1)_{j}} = B_{j,n-k} \frac{(n-k-j)^{4}}{(n-k+j)^{4}}$$

$$\dots$$

$$B_{j,n-k-2i} = \frac{(-n+k+2i)_{j}(n-k-2i+1)_{j}}{(1)_{j}} = B_{j,n-k} \frac{(n-k-j)^{2i}}{(n-k+j)^{2i}}$$

In the previous equation, the symbol $(x)^{\underline{n}}$ is used to represent the falling factorial. From equations (26) and identities (27), a recurrence relation for the coefficients α_{n-k-2i} can be derived. It can be shown that:

$$\alpha_{n-k-2i} = \frac{2^{k+2i}(n-1/2)\underline{k}(n-i)\underline{i}(n-k-1/2)\underline{2i}}{(2i)\underline{2i}(n-1/2)\underline{i}} - \sum_{l=0}^{i-1} \frac{(2(n-k-i-l))\underline{2(i-l)}}{(2(i-l))\underline{2(i-l)}} \alpha_{n-k-2l}.$$
(28)

3. Conclusion

In this work, we have studied the scaling properties of Legendre polynomials $P_n(x)$. We have demonstrated that $P_n(\lambda x)$, where λ is a constant, can be expanded either as a sum of Legendre polynomials $P_n(x)$ or their multiple derivatives, and we have obtained an explicit expression for the expansion coefficients. In addition, we have shown that the multiple derivative $d^k P_n(x)/dx^k$ can also be expressed as a sum of Legendre polynomials and we derived a recurrence relation for the coefficients.

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References

- Gradshteyn I.F. and Ryzhik I.M., Table of Integrals, Series, and Products 8th Edition: Academic Press; 2014.
- [2] Olver F.W.J. and Lozier D.W. and Boisvert R.F. and Clark C.W., NIST Handbook of Mathematical Functions - Cambridge University Press; 2010.
- [3] Bosch W., On the Computation of Derivatives of Legendre Functions, Physics and Chemistry of the Earth (A), 25, 655 (2000).

- [4] Anli F. and Gungor S., Some useful properties of Legendre polynomials and its applications to neutron transport equation in slab geometry, Applied Mathematical Modelling, 31, 727 (2007).
- [5] Antonov V.A. and Kholshevnikov K.V. and Shaidulin V.Sh., Estimating the Derivative of the Legendre Polynomial, Vestnik St. Petersburg University. Mathematics, 43, 191 (2010).
- [6] Szmytkowski R., On the derivative of the associated Legendre function of the first kind of integer order with respect to its degree (with applications to the construction of the associated Legendre function of the second kind of integer degree and order), Journal of Mathematical Chemistry, 49, 1436 (2011).
- [7] Dattoli G. and Germano B. and Martinelli M.R. and Ricci P.E., A novel theory of Legendre polynomials, Mathematical and Computer Modelling, 54, 80 (2011).
- [8] Bos L. and Narayan A. and Levenberg N. and Piazzon F., An Orthogonality Property of the Legendre Polynomials, Constructive Approximation, **45**, 65 (2017).
- [9] Polyanin A.D. and Manzhirov A.V., Handbook of Mathematics for Engineers and Scientists: Chapman & Hall/CRC; 2007.