King Saud University
Department of Mathematics

Final Exam in Math 151, S1-1447H.

(This is a two-page long exam)
Calculators are not allowed

Q1. **(a)** Without using truth tables, show that $(p \to q) \wedge (q \vee r)$ is logically equivalent to $(p \vee \neg r) \to q$. (3)

(b) Use induction to prove the following for every $n \geq 1$:

$$3 + 9 + 15 + \cdots + (6n - 3) = 3n^2. \ (4)$$

(c) Use a direct proof to show that if m is an integer such that $m+3$ is even, then $m^2 - 2m + 5$ is divisible by 4. (3)

Q2. **(a)** Let $R = \{(1,1), (1,4), (2,2), (3,1), (4,1), (4,4)\}$ be a relation on $A = \{1,2,3,4\}$. Determine whether R is reflexive, symmetric, antisymmetric or transitive. (Justify your answers).(4)

(b) Let E be the relation on $\mathbb{Z} - \{0\}$ defined by $aEb \Leftrightarrow a^3 b > 0$.

(i) Prove that E is an equivalence relation. (3)

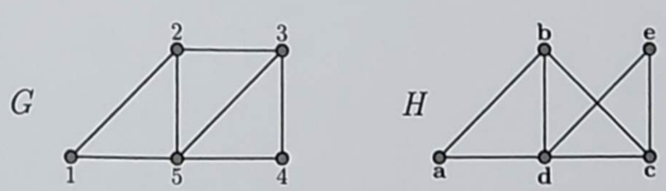
(ii) Find $[1]$ and $[-1]$. (2)

(iii) What is the number of (distinct) equivalence classes? (Justify your answer.) (1)

Q3. **(a)** (i) Find the number of edges of the complement $\overline{K_{6,8}}$ of $K_{6,8}$. (1)

(ii) Is $\overline{K_{6,8}}$ isomorphic to $K_{6,8}$? (Justify your answer.) (1)

(b) Determine whether the following graphs G and H are isomorphic. (2)



(c) (i) Is the complement of every connected simple graph itself connected? (Justify your answer.) (1)

(ii) Is there a bipartite graph which is 0-regular? (Justify your answer.) (1)

Q4.(a) Let T be a tree having 6 vertices of degree 1, 3 vertices of degree 2 and 2 vertices of degree x.

(i) Find the number of edges of T. (1)

(ii) Find the value of x. (2)

(b) For the graph G in **Q3(b)**, find a spanning tree with root 1,

(i) using *depth-first* search; (1)

(ii) using *breadth-first* search. (1)

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(c) Using alphabetical order, form a binary search tree for the words:
Iron, Copper, Silver, Gold, Nickel, Tin, Lead. (2)

Q5. **(a)** For the Boolean function $f(x, y, z) = (x\overline{z} + y)(x\overline{y} + y)$, find

(i) the complete sum-of-products expansion (CSP); (2)

(ii) the complete product-of-sums expansion (CPS). (2)

(b) Let $g(x, y, z) = xyz + x\,\overline{y}z + \overline{x}\,y\overline{z} + \overline{x}\,\overline{y}\,\overline{z} + \overline{x}\,\overline{y}z$ be a Boolean function.

(i) Build the Karnaugh map of g. (1)

(ii) Simplify g (i.e., write it in MSP form). (2)

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Q1) 10

a) $(p \to q) \wedge (q \vee r) \equiv (\neg p \vee q) \wedge (q \vee r)$

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$\equiv q \vee (\neg p \wedge r)$

$\equiv \neg(p \vee \neg r) \vee q$

$\equiv (p \vee \neg q) \to q.$

b) $P(n): 3+9+15+\dots+(6n-3) = 3n^2;\ n \geq 1.$

B.S: $P(1): 3 = 3\times 1^2 = 3$ True.

IS: Let $k \geq 1$; we prove that $P(k) \longrightarrow P(k+1)$.

assume that $P(k)$ is true:

4

$3+9+15+(6k-3) = 3k^2;$

we show that $P(k+1)$ is true:

$3+9+15+(6k+3) = 3(k+1)^2$??

$3+9+15+\dots+(6k-3)+(6k+3) = 3k^2+6k+3$

$= 3(k^2+2k+1)$

$= 3(k+1)^2.$

$\Rightarrow P(k+1)$ is true

$\Rightarrow \forall n \geq 1;\ P(n)$ is true.

c) let $m \in \mathbb{Z}$.
Assume that $m+3$ is even $\Rightarrow m+3 = 2t;\ t \in \mathbb{Z}.$

$\Rightarrow m = 2t-3.$

$m^2-2m+5 = (2t-3)^2 - 2(2t-3)+5$

$= 4t^2-12t+9-4t+6+5$

$= 4t^2-16t+20 = 4(\underbrace{t^2-4t+5}_{\in \mathbb{Z}})$

$\Rightarrow 4 \mid (m^2-2m+5).$

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Q2) a) 10

- $(3,3) \notin R \Rightarrow R$ is not reflexive. 1
- $(3,1) \in R$; $(1,3) \notin R \Rightarrow R$ is not symmetric 1
- $(1,4) \in R$; $(4,1) \notin R \Rightarrow R$ is not anti-symmetric 1
- $(3,1) \in R$; $(1,4) \in R$, but $(3,4) \notin R$, $\Rightarrow R$ is not transitive 1

b) i)

- $a \in \mathbb{Z}-\{0\}$; $a^3 a = a^4 > 0 \Rightarrow a E a \Rightarrow E$ is reflexive 1
- $a, b \in \mathbb{Z}-\{0\}$; $a E b \Rightarrow a^3 b > 0 \overset{\times b^2}{\Rightarrow} a^3 \cdot b \cdot b^2 > 0$

$\Rightarrow a^3 b^3 > 0$

$\overset{\div a^2}{\Rightarrow} \frac{a^3 b^3}{a^2} > 0 \Rightarrow a b^3 > 0$

$\Rightarrow b^3 a > 0 \Rightarrow b E a$

$\Rightarrow E$ is symmetric 1

- $a, b, c \in \mathbb{Z}-\{0\}$; $a E b$ and $b E c \Rightarrow a^3 b > 0$ and $b^3 c > 0$

$\Rightarrow a^3 b \cdot b^3 c > 0$

$\Rightarrow a^3 b^4 c > 0$

$\Rightarrow a^3 c > 0 \Rightarrow a E c$

$\Rightarrow E$ is transitive 1

ii) $[1] = \{a \in \mathbb{Z}-\{0\}; a E 1\} = \{a \in \mathbb{Z}-\{0\}; a^3 > 0\} = \mathbb{Z}^+$. 1

$[-1] = \{a \in \mathbb{Z}-\{0\}; a E (-1)\} = \{a \in \mathbb{Z}-\{0\}; -a^3 > 0\}$

$= \{a \in \mathbb{Z}-\{0\} \; a^3 < 0\} = \mathbb{Z}^-$ 1

iii) The number of distinct equivalence classes is 2.
$\mathbb{Z}^+$ and $\mathbb{Z}^-$; because $\mathbb{Z}^+$ and $\mathbb{Z}^-$ form a partition of $\mathbb{Z}-\{0\}$

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$d_3)$ (6)

a) i) $K_{6,8} \to |V| = 14$, $|E| = 48$

(1)

$$|E(K_{6,8})| + |E(\overline{K_{6,8}})| = \frac{|V|(|V|-1)}{2} = \frac{14 \times 13}{2} = 91$$

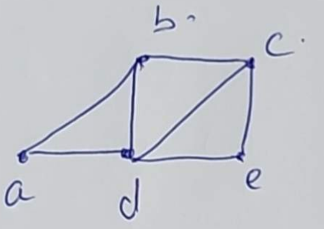
$$\Rightarrow |E(\overline{K_{6,8}})| = 91 - 48 = 43$$

ii) $|E(K_{6,8})| \neq |E(\overline{K_{6,8}})| \Rightarrow K_{6,8}$ is not isomorphic to $\overline{K_{6,8}}$ (1)

b) G and H are isomorphic.

R1)

x	1	2	5	4	3
$f(x)$	a	b	d	e	c

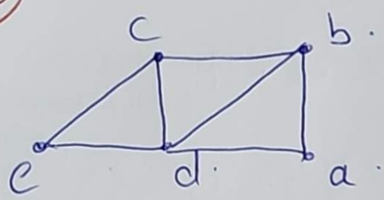


H

(2)

R2)

x	1	2	5	3	4
$f(x)$	e	c	d	b	a

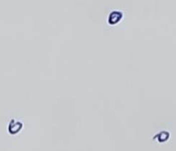


H

c) i) No;

(1)

G is connected



$\bar{G}$ is disconnected.

ii) yes, the empty graph is bipartite and 0-regular.

(1)

Q4) 7

a) i) $|V(T)| = 11 \Rightarrow |E(T)| = 10 = |V(T)| - 1$ 1

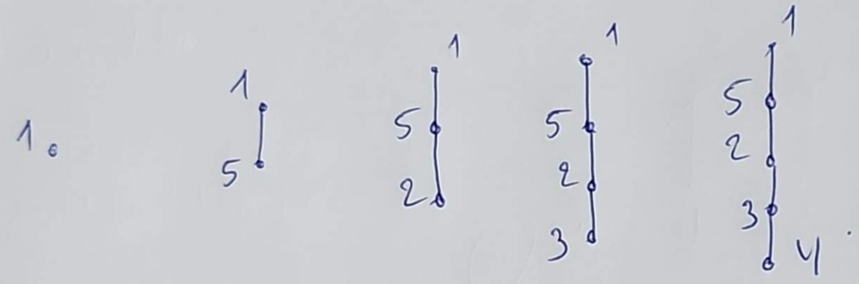
ii) $6\times1 + 3\times2 + x\times2 = |E(T)|\times 2$

$\Rightarrow 6+6+2x = 20$

$\Rightarrow 2x = 20-12 = 8$ 2

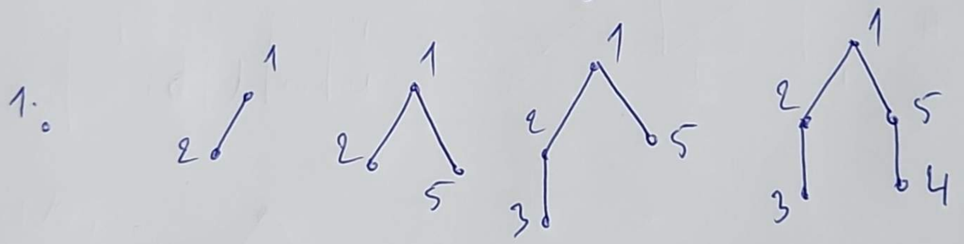
$\Rightarrow x = 4$

b) i)



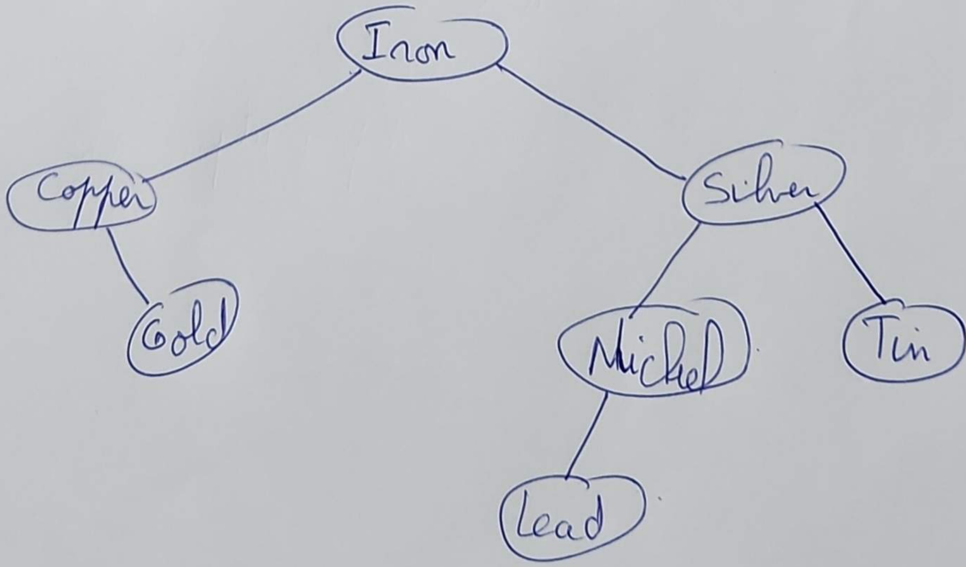
G_0 G_1 G_2 G_3 G_4 1

ii)



G_0 G_1 G_2 G_3 G_4.

c). 2



4)

Q5) 7

a) i) $f(x,y,z) = (x\bar{z} + y)(x\bar{y} + y)$

$= x\bar{y}\bar{z} + xy\bar{z} + 0 + y.$

2 $= x\bar{y}\bar{z} + xy\bar{z} + xyz + xy\bar{z} + \bar{x}yz + \bar{x}y\bar{z}$

$= x\bar{y}\bar{z} + xy\bar{z} + xyz + \bar{x}yz + \bar{x}y\bar{z} = CSP(f(x,y,z)).$

ii) $\bar{f}(x,y,z) = (\bar{x}+z)\cdot\bar{y} + (\bar{x}+y)\cdot\bar{y}$

$= \bar{x}\bar{y} + \bar{y}z + \bar{x}\bar{y} + 0$

$= \bar{x}\bar{y}z + \cancel{\bar{x}\bar{y}\bar{z}} + x\bar{y}z + \cancel{\bar{x}\bar{y}z} + \cancel{\bar{x}\bar{y}z} + \bar{x}\bar{y}\bar{z}.$ 2

$= \bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z} + x\bar{y}z.$

$\Rightarrow CPS(\bar{f}) = (x+y+\bar{z})(x+y+z)(\bar{x}+y+\bar{z}).$

b) i)

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	1			1
$\bar{x}$		1	1	1

1

ii) $MSP(g) = xz + \bar{x}\bar{z} + \bar{y}z.$ 2

5)