King Saud University
Department of Mathematics

Final Exam in Math 151, S2-1445H.

(This is a two page long exam)
Calculators are not allowed

Q1. (a) Without using truth tables, show that $\neg(p \wedge q) \wedge (p \to \neg r)$ is logically equivalent to $(q \vee r) \to \neg p$. (3)

(b) Use induction to prove that $9 + 13 + 17 + \ldots + (4n+5) = n(2n+7)$ for all $n \geq 1$. (4)

(c) For any integers x and y, prove by contraposition that: if $x^2(y+3)$ is even then x is even or y is odd. (2)

Q2. (a) Let A be the set of **even** integers, and let E be the relation on A defined by aEb if and only if 4 divides $a+b$.

(i) Show that E is an equivalence relation. (3)
(ii) Find $[0]$. (1)
(iii) Is -6 related to 12? (Justify your answer.) (1)

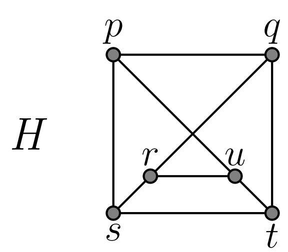
(b) Let $P = \{(1,1),(1,3),(1,4),(2,2),(3,3),(3,4),(4,4)\}$ be a relation on $B = \{1,2,3,4\}$.

(i) Show that P is a partial ordering. (3)
(ii) Is P a total ordering? (Justify your answer.) (1)
(iii) Represent P with a Hasse diagram. (1)

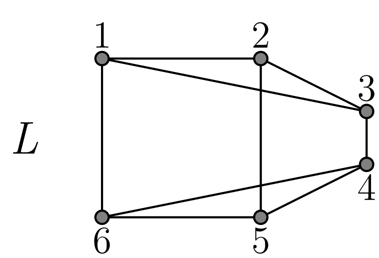
Q3. (a) Let G be a graph with 14 edges and degree sequence $a, a, 5, 5, 5, 5$.

(i) Find a. (1)
(ii) Can G be a complete graph? (Justify your answer.) (1)

(b) Determine whether the following graph H is bipartite, and if so, then find a bipartite representation. (2)



(c) Determine whether the graph H in **(b)** is isomorphic the graph L below. (Justify your answer.) (2)



(d) Let M be a (undirected) graph represented with the following adjacency matrix.

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Draw M, and show that it is not a tree. (2)

(e) For the graph H in **(b)**, find a spanning tree with root r,

(i) using *depth-first* search; (1)

(ii) using *breadth-first* search. (1)

(f) Using alphabetical order, form a binary search tree for the words:
Saw, Hammer, Screwdriver, Ax, Pliers, Wrench, Drill. (2)

Q4. (a) Without using tables, prove the following Boolean identity:

$$\overline{\overline{(\overline{x} + y)} + z} = \overline{x}\,\overline{z} + y\overline{z}. \text{ (2)}$$

(b) (i) Find the complete sum-of-products expansion (CSP) for
$f(x, y, z) = (x\overline{z} + y)(x + yz)$. (2)

(ii) Find the complete product-of-sums expansion (CPS) for
$g(x, y, z) = \overline{x}z + y$. (2)

(c) Let $h(x, y, z) = xyz + xy\overline{z} + x\overline{y}z + \overline{x}yz + \overline{x}\,\overline{y}\,\overline{z} + \overline{x}\,\overline{y}z$ be a Boolean function.

(i) Build the Karnaugh map of h. (1)

(ii) Simplify h (i.e., write it in MSP form). (2)

جامعة الملك سعود
King Saud University

كلية العلوم
قسم الرياضيات

College of Science.
Department of Mathematics

Final Exam
Academic Year 1445 Hijri- SecondSemester

Exam Information معلومات الامتحان			
Course name	Discrete Mathematics الرياضيات المحددة		اسم المقرر
Course Code	151 ريض		رمز المقرر
Exam Date	2024-06-03	1445-11-26	تاريخ الامتحان
Exam Time	08: 00 AM		وقت الامتحان
Exam Duration	3 hours	ثلاث ساعات	مدة الامتحان
Classroom No.			رقم قاعة الاختبار
Instructor Name			اسم استاذ المقرر

Student Information معلومات الطالب		
Student's Name		اسم الطالب
ID number		الرقم الجامعي
Section No.		رقم الشعبة
Serial Number		الرقم التسلسلي

General Instructions: / **تعليمات عامة:**

Your Exam consists of 9 PAGES (except this paper)
عدد صفحات الامتحان 9 صفحة. (باستثناء هذه الورقة)

Keep your mobile and smart watch out of the classroom.
يجب إبقاء الهواتف والساعات الذكية خارج قاعة الامتحان.

Calculators are not allowed.
الآلة الحاسبة ممنوعة.

هذا الجزء خاص بأستاذ المادة

This section is ONLY for instructor

Course Learning Outcomes (CLOs)		Related Question (s)	Points	Final Score
.1		2a(ii,iii)		
.2		2b(iii), 4b		
.1		1, 3(e,f), 4a		
2		2a(i), 2b(i,ii)		
3		3(a to d)		
4		4c		

Q	1	2ai,ii	2aiii	2bi,ii	2biii	3a-d	3e,f	4a	4b	4c	Total
Grade											

Q1. (a) Without using truth tables, show that $\neg(p \wedge q) \wedge (p \rightarrow \neg r)$ is logically equivalent to $(q \vee r) \rightarrow \neg p$. (3 points)

$\neg(p\wedge q)\wedge(p\rightarrow\neg r) \equiv (\neg p \vee \neg q)\wedge(\neg p \vee \neg r)$ DeMorgan's law and $p \rightarrow q \equiv \neg$

$\equiv \neg p \vee (\neg q \wedge \neg r)$ Distributive law.

$\equiv \neg p \vee \neg(q \vee r)$ DeMorgan's law

$\equiv \neg(q\vee r) \vee \neg p$ commutitive law

$\equiv (q \vee r) \rightarrow \neg p$ as $p \rightarrow q \equiv \neg p \vee q$.

$\therefore \neg(p\wedge q)\wedge(p\rightarrow\neg r) \equiv (q\vee r)\rightarrow \neg p$.

(b) For any integers x and y, prove by contraposition that: if $x^2(y+3)$ is even then x is even or y is odd. (2 points)

We want to prove:

If x is odd and y is even then $x^2(y+3)$ is odd.

Proof: Let x be odd and y even. $\exists k, t \in \mathbb{Z}$ s.t

$x = 2k+1$, $y = 2t$

Now $x^2(y+3) = (2k+1)^2(2t+3)$

$= (4k^2+4k+1)(2t+3)$

$= 8k^2t + 12k^2 + 8kt + 12k + 2t + 3$

$= 2(4k^2t + 6k^2 + 4kt + 6k + t + 1) + 1$

$= 2s+1$

put $4k^2t + 6k^2 + 4kt + 6k + t + 1 = s$ for some $s \in \mathbb{Z}$

$\therefore x^2(y+3)$ is odd.

We proved if x is odd and y is even then $x^2(y+3)$ is odd, which is equivalent to if $x^2(y+3)$ is even then x is even or y is odd.

(c) Use induction to prove that $9 + 13 + 17 + \ldots + (4n + 5) = n(2n + 7)$ for all $n \geq 1$. (4 points)

$P(n): 9+13+17+\ldots+(4n+5) = n(2n+7)$.

1. Basis Step: $n=1$.

$L.H.S = 4(1)+5 = 9$, $R.H.S = 1(2 \cdot 1+7) = 9$

$L.H.S = R.H.S$ $\therefore$ True for $n=1$.

2. Inductive Step: Suppose $P(k)$ is true for

$9+13+17+\ldots+(4k+5) = k(2k+7)$ ✱ I.H.

We want to prove that $P(k+1)$ is true,

$9+13+17+\ldots+(4(k+1)+5) \overset{??}{=} (k+1)[2(k+1)+7]$.

that is: $9+13+17+\ldots+(4k+9) \overset{??}{=} (k+1)(2k+9)$.

$L.H.S = 9+13+17+\ldots+(4k+5)+(4k+9) = k(2k+7)+(4k+9)$

$= 2k^2+7k+4k+9$

$= 2k^2+11k+9$

$= (2k+9)(k+1)$

$= R.H.S.$

$\therefore$ $P(k+1)$ is true.

From pricipal of Mathematical Induction, $P(n)$ true $\forall n \geq 1$.

Q2. **(a)** Let A be the set of **even** integers, and let E be the relation on A defined by aEb if and only if 4 divides $a + b$.

(i) Show that E is an equivalence relation. (3 points)

$aEb \leftrightarrow 4 \mid a+b$.

① $\forall a \in A$, $a+a=4k$, as $a \in A$, $\exists k \in \mathbb{Z}$ s.t $a=2k$.

$\therefore 4 \mid a+a$

That $\therefore$ E is reflexive.

② If aEb then $a+b=4k$ for some $k \in \mathbb{Z}$.

but $b+a=a+b=4k$, $\therefore bEa$. That is

$\therefore \forall a,b \in A$ if aEb then bEa.

E is symmetric.

③ If $aEb \wedge bEc$, then $\exists k,t \in \mathbb{Z}$ s.t

$a+b=4k \wedge b+c=4t$.

$a+b+(b+c)=4k+4t$.

Now $(a+b)+(b+c)=4k+4t$, but $b \in A$ That is $b=2s$ for some $s \in \mathbb{Z}$

$a+c = 4k+4t+2b$

$= 4k+4t+4s$ put $k+t+s=y$ for some $y \in \mathbb{Z}$

$\therefore a+c = 4(k+t+s)$

$\therefore a+c=4y$.

$\therefore \forall a,b,c$ if $aEb \wedge bEc$ Then aEc and E is transitive. From ①, ②, ③ E is an equivalence relation.

(ii) Find [0]. (1 point)

$[0]=\{s \in A \mid (s,0) \in E\} = \{s \in A \mid s+0=4k$, for some $k \in \mathbb{Z}\}$

$=\{s \in A \mid s=4k\}$

$=\{\ldots, -12, -8, -4, 0, 4, 8, 12, \ldots\}$.

(iii) Is −6 related to 12? (Justify your answer.) (1 point)

No, as $-6+12=6$ and $4 \nmid 6$.

(b) Let $P = \{(1,1),(1,3),(1,4),(2,2),(3,3),(3,4),(4,4)\}$ be a relation on $B = \{1,2,3,4\}$.

(i) Show that P is a partial ordering. (3 points)

① P is reflexive, as $\forall a \in B$, $(a,a) \in P$.
$(1,1), (2,2), (3,3), (4,4) \in P$.

② P is antisymmetric as $(1,3) \in P$ but $(3,1) \notin P$
$(1,4) \in P$ but $(4,1) \notin P$
$(3,4) \in P$ but $(4,3) \notin P$

$\therefore \forall a,b \in B$, if $(a,b) \in P \wedge (b,a) \in P$ then $a=b$.

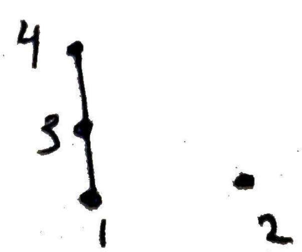
③ P is transitive,
$P^2 = \{(1,1),(1,3),(1,4),(2,2),(3,3),(4,4)\} \subseteq P$
$P^3 = \{(1,1),(1,3),(1,4),(2,2),(3,3),(4,4)\} = P^2 \subseteq P$
$P^n = P^2 \ \forall n \geq 2$.
$\therefore P$ is transitive as $P^n \subseteq P$.

From ①, ②, ③ P is a partial ordering.

(ii) Is P a total ordering? (Justify your answer.) (1 point)

NO, as $2 \not P 3$ and $3 \not P 2$.

(iii) Represent P with a Hasse diagram. (1 point)



Q3. **(a)** Let G be a graph with 14 edges and degree sequence $a, a, 5, 5, 5, 5$.
(i) Find a. (1 point)

We know that $2e = \sum_{v \in V} \deg(v)$

$2(14) = a + a + 5+5+5+5 = 2a + 20$

$28 - 20 = 2a \rightarrow 8 = 2a$

$\therefore a = 4$.

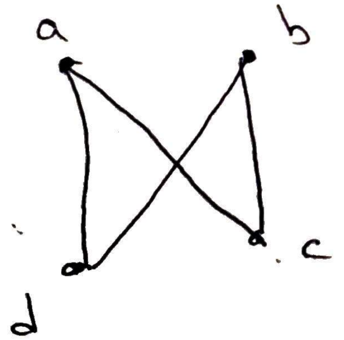
(ii) Can G be a complete graph? (Justify your answer.) (1 point)

No, as a complete graph with n vertices has $\frac{n(n-1)}{2}$ edges.
In this graph we have 6 verticies so, to be complete it must have $\frac{6(5)}{2} = 15$ edges which is not the case.

(b) Let M be a (undirected) graph represented with the following adjacency matrix.

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

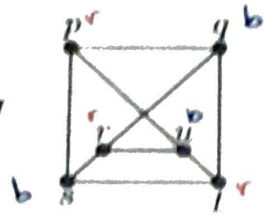
Draw M, and show that it is not a tree. (2 points)



The graph M is not a tree as it has a simple circuit.

(c) Determine whether the following graph H is bipartite, and if so, then find a bipartite representation. (2 points)

H



The graph H is bipartite.

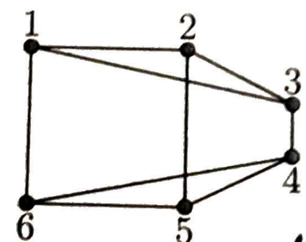
$\phi \neq V_1 = \{p, r, t\}$, $\phi \neq V_2 = \{q, u, s\}$, $V_1 \cap V_2 = \phi$

$V_1 \cup V_2 = V = \{p, q, r, s, t, u\}$.

The bipartite representation is (V_1, V_2).

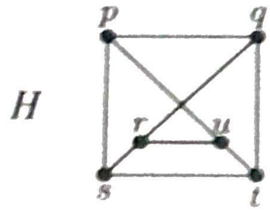
(d) Determine whether the graph H in **(c)** is isomorphic the graph L below. (Justify your answer.) (2 points)

L

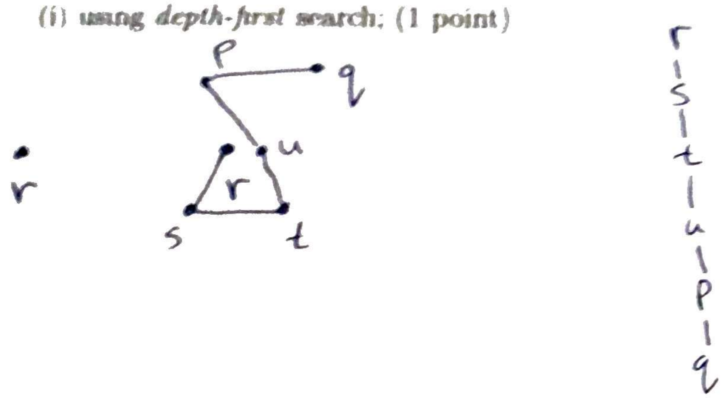


Although H and L both have 6 verticies and 8 edges and each vertex is of degree 3, they are not isomorphic. As H is bipartite wheras L is not. Also H has a simple circuit of length 4 wheras L has a simple circuit of length 3.

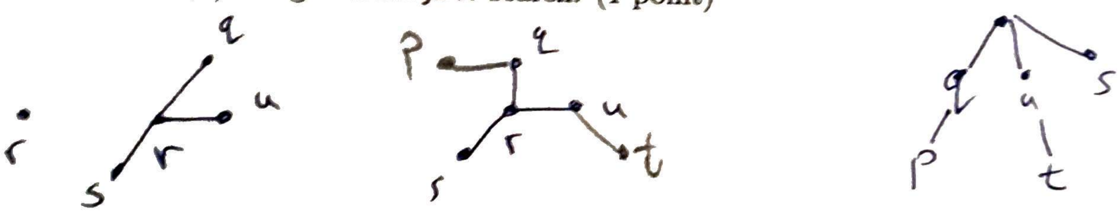
(e) For the graph H below, find a spanning tree with root r.



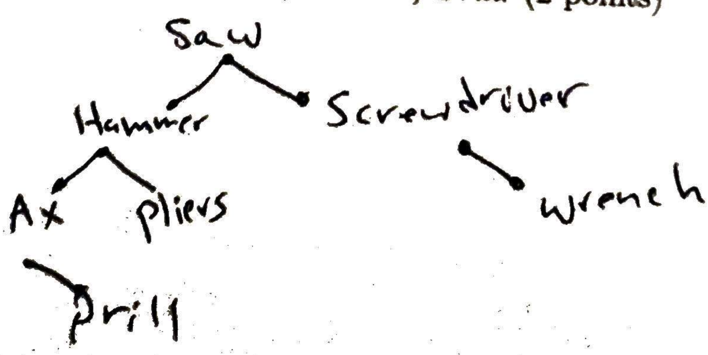
(i) using *depth-first* search; (1 point)



(ii) using *breadth-first* search. (1 ponit)



(f) Using alphabetical order, form a binary search tree for the words: *Saw, Hammer, Screwdriver, Ax, Pliers, Wrench, Drill.* (2 points)



Q4. (a) Without using tables, prove the following Boolean identity:

$$\overline{\overline{(\overline{x} + y)} + z} = \overline{x}\,\overline{z} + y\overline{z}. \text{ (2 points)}$$

$$L.H.S = \overline{\overline{(\overline{x}+y)}+z} = \overline{\overline{(\overline{x}+y)}} \cdot \overline{z}$$ De Morgan's law

$$= (\overline{x}+y) \cdot \overline{z}$$ Double negation

$$= \overline{x}\,\overline{z} + y\,\overline{z}$$

$$= R.H.S$$

$$\therefore \overline{\overline{(\overline{x}+y)}+z} = \overline{x}\,\overline{z} + y\overline{z}.$$

(b) (i) Find the complete sum-of-products expansion (CSP) for $f(x, y, z) = (x\overline{z} + y)(x + yz)$. (2 points)

$$f(x,y,z) = xx\overline{z} + xy + xy + yyz$$

$$= x\overline{z} + xy + yz.$$

$$CSP(f) = \cancel{x}\overline{z} + xy + yz$$

$$= x(y+\overline{y})\overline{z} + xy(z+\overline{z}) + (x+\overline{x})yz$$

$$= xy\overline{z} + x\overline{y}\overline{z} + xyz + xy\overline{z} + xyz + \overline{x}yz$$

$$= xy\overline{z} + x\overline{y}\overline{z} + xyz + \overline{x}yz.$$

(ii) Find the complete product-of-sums expansion (CPS) for $g(x, y, z) = \overline{x}z + y$. (2 points)

$CPS(g) = [CSP(g^d)]^d$.

$g^d(x,y,z) = (\bar{x}+z)y = \bar{x}y + yz$.

$csp(g^d) = \bar{x}y(z+\bar{z}) + (x+\bar{x})yz$

$= \bar{x}yz + \bar{x}y\bar{z} + xyz + \bar{x}yz$

$= \bar{x}yz + \bar{x}y\bar{z} + xyz$

$[csp(g^d)]^d = (\bar{x}+y+z)(\bar{x}+y+\bar{z})(x+y+z)$

$= cps(g)$.

(c) Let $h(x, y, z) = xyz + xy\overline{z} + x\overline{y}z + \overline{x}yz + \overline{x}\,\overline{y}\,\overline{z} + \overline{x}\,\overline{y}z$ be a Boolean function.

(i) Build the Karnaugh map of h. (1 point)

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	1	1		1
$\bar{x}$	1		1	1

(ii) Simplify h (i.e., write it in MSP form). (2 points)

$z + xy + \bar{x}\bar{y}$

Good Luck :)