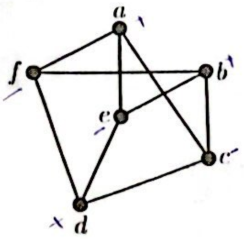
King Saud University
Department of Mathematics

Final Exam in Math 151, T3-1444H.
(The exam is 2-pages long)
Calculators are definitely not allowed

Q1. (a) Without using truth tables show that $[p \wedge (\neg q \rightarrow p)] \rightarrow p$ is a tautology. [3]
(b) Use induction to show that $6 + 18 + 36 + \ldots + 3n(n+1) = n(n+1)(n+2)$ for all $n \geq 1$. [4]
(c) Suppose m, n and l are integers. Use contraposition to prove that if $l \nmid (mn + l^2)$, then $l \nmid n$. [2]

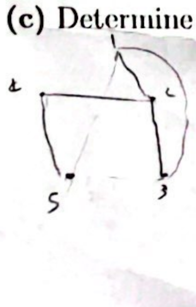
Q2. (a) Define a relation E on $\mathbb{Z}$ as mEn if and only if $m - n$ is even.
(i) Prove that E is an equivalence relation. [3]
(ii) Show that $(m, -m) \in E$ for every $m \in \mathbb{Z}$, and that $[0] \neq [1]$. [2]
(b) Define a relation P on $A = \{1, 2, 3, 6\}$ as aPb if and only if $a \mid (a + b)$.
(i) List all ordered pairs of P. [2]
(ii) Show that P is a partial order. [3]
(iii) Is P a total order? (Justify your answer.) [1]
(iv) Represent P by a Hasse diagram. [1]

Q3. (a) For any integer $n \geq 2$, show that there exists no complete graph K_n with n vertices and $n^2 - 1$ edges . [2]
(b) (i) Determine whether the graph G below is bipartite. If so, find a bipartite representation. [2]
(ii) Represent G with an adjacency matrix. [1]

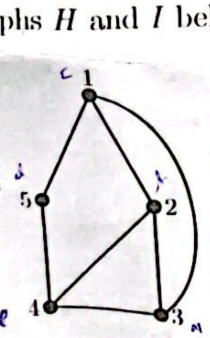


G

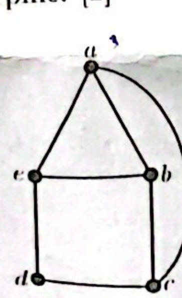
(c) Determine if the graphs H and I below are isomorphic. [2]



H



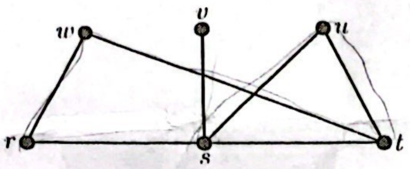
I



Q4. **(a)** For the graph J below, find a spanning tree with root r,

(i) using *depth-first* search; [1]

(ii) using *breadth-first* search. [1]



J

(b) If T is a tree with degree-sequence $4, 3, 2, x, x, x, x, x$, then find the value of x. [1]

(c) Using alphabetical order, form a binary search tree for the words:
Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune. [2]

Q5. **(a)** Let $f(x, y, z) = x\overline{y} + z$ be a Boolean function.

(i) Find the complete sum-of-products expansion (CSP) of f. [2]

(ii) Find the complete product-of-sums expansion (CPS) of f. [2]

(b) Let $g(x, y, z) = xyz + xy\overline{z} + x\overline{y}\,\overline{z} + x\overline{y}z + \overline{x}yz + \overline{x}y\overline{z} + \overline{x}\,\overline{y}z$ be a Boolean function.

(i) Build the Karnaugh map of g. [1]

(ii) Simplify g (i.e., write it in MSP form). [2]

Final 44 S3

Q1. (a) Without using truth tables show that $[p \wedge (\neg q \rightarrow p)] \rightarrow p$ is a tautology. [3]

$\equiv [p \wedge (\neg\neg q \vee p)] \rightarrow p$ as $p \rightarrow q \equiv \neg p \vee q$

$\equiv [p \wedge (q \vee p)] \rightarrow p$ double negation law

$\equiv [p \wedge (p \vee q)] \rightarrow p$ commutative law

$\equiv p \rightarrow p$ absorption law

$\equiv \neg p \vee p$ as $p \rightarrow q \equiv \neg p \vee q$

$\equiv T$ negation law

tautology

(b) Use induction to show that $6 + 18 + 36 + \ldots + 3n(n+1) = n(n+1)(n+2)$ for all $n \geq 1$. [4]

$P(n)$

1) Basis step at $n = 1$

$3(1)(1+1) \stackrel{?}{=} 1(1+1)(1+2)$

$3 \times 2 \stackrel{?}{=} 2 \times 3$

$6 = 6$

so $p(1)$ is true

2) inductive step $p(k) \rightarrow p(k+1)$

assume $p(k)$ is true for same $k \geq 1$

$P(K)$: $6+18+36+ \dots +3K(K+1) = K(K+1)(K+2)$ * I.H

we want to prove

$P(K+1)$: $\underbrace{6+18+36+ \dots + 3K(K+1)}_{P(K)} + 3(K+1)(K+1+1) \overset{??}{=} (K+1)(K+2)(K+3)$

L.H.S : $K(K+1)(K+2) + 3(K+1)(K+2)$

$\underbrace{(K+1)(K+2)}_{\text{common factor}}\underbrace{(K+3)}_{R.H.S}$

common factor

R.H.S

s P(K+1) is true, we proved $P(K) \to P(K+1)$ is true
using M.I, we proved that p(n) is true

$\forall n \geq 1$

(c) Suppose m, n and l are integers. Use contraposition to prove that if $l \nmid (mn + l^2)$, then $l \nmid n$. [2]

contraposition:

$\to mn+l^2 = lt$

prove: $l \mid n \to l \mid (mn+l^2)$

Suppose $l \mid n$ is true, $\exists K \in \mathbb{Z}$ s.t. $n = lK$

$mn + l^2 = m(lK) + l^2$

$= l(mK + l)$, say $mK+l = t$, $t \in \mathbb{Z}$

$\therefore mn+l^2 = lt$

$\therefore l \mid mn+l^2$, we proved $l \mid n \to l \mid (mn+l^2)$

so by contraposition, $l \nmid (mn+l^2) \to l \nmid n$

Q2. (a) Define a relation E on $\mathbb{Z}$ as mEn if and only if $m-n$ is even.
(i) Prove that E is an equivalence relation. [3]
(ii) Show that $(m,-m) \in E$ for every $m \in \mathbb{Z}$, and that $[0] \neq [1]$. [2]

i) to prove it's equivalence we need to prove its

reflexive, symmetric, transitive

1) $\forall a$, $a-a=0$

0 is even

so the relation E is reflexive

2) $\forall a \forall b$ $\quad aEb \overset{?}{\longrightarrow} bEa$

aEb : $a-b$ is even, so $\exists k \in \mathbb{Z}$ s.t

$a-b=2k$

$b-a=-2k$

$b-a=2(-k)$, $-k \in \mathbb{Z}$

$\therefore bEa$

$\therefore$ relation E is symmetric

3) $\forall a \forall b \forall c$, $\quad aEb \wedge bEc \overset{?}{\longrightarrow} aEc$

aEb : $a-b$ is even, $\exists k \in \mathbb{Z}$ s.t

$a-b=2k$

bEc : $b-c$ is even, $\exists t \in \mathbb{Z}$ s.t

$b-c=2t$

$(a-b)+(b-c)=2k+2t$

$a-c = 2(k+t)$, put $k+t=m$, $m \in \mathbb{Z}$

$a-c=2m$

$\therefore a-c$ is even

$\therefore aEc$

$\therefore$ E is transitive

from ① ② ③ E is an equivalence relation

ii) $(m, -m) \overset{?}{\in} E$

$m - (-m) = m + m = 2m$ even

so $(m, -m) \in E$ for every $m \in \mathbb{Z}$

$[0] \overset{?}{\neq} [1]$

Check if $0 \overset{?}{R} 1$ or $1 \overset{?}{R} 0$

$0R1$: $0 - 1 = -1$ } odd so $0 \not R 1$

$1R0$: $1 - 0 = 1$ } odd so $1 \not R 0$

so $[0] \neq [1]$

(b) Define a relation P on $A = \{1, 2, 3, 6\}$ as aPb if and only if $a \mid (a + b)$.

(i) List all ordered pairs of P. [2]

(ii) Show that P is a partial order. [3]

(iii) Is P a total order? (Justify your answer.) [1]

(iv) Represent P by a Hasse diagram. [1]

$a + b = aK,\ \exists K \in \mathbb{Z}$

i) $P = \{(1,1), (1,2), (1,3), (1,6), (2,2), (2,6), (3,3), (3,6), (6,6)\}$

ii) to show its partially ordered we need to prove that P is reflexive, antisymmetric, and transitive

1) reflexive since $\forall a \in A,\ (a,a) \in P$

2) antisymmetric: $\forall a \forall b \in A,\ (a,b) \in P \wedge (b,a) \in P \rightarrow a = b$

hypothesis is false as $(a,b) \in P$ but $(b,a) \notin P$

$T \wedge F \equiv F$

$F \rightarrow q \equiv T$

so P is antisymmetric

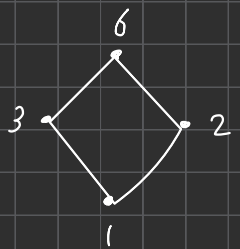
3) transitive since $\forall a \forall b \forall c \in P$, $(a,b) \in P \land (b,c) \in P \rightarrow (a,c) \in P$

from ① ② ③ P is a partial order

iii) no as not every element in A is comparable

ex: $2 \not P 3$ and $3 \not P 2$

iv)



Q3. (a) For any integer $n \geq 2$, show that there exists no complete graph K_n with n vertices and $n^2 - 1$ edges . [2]

$|E|$

in a complete graph:

$$|E| = \frac{n(n-1)}{2}$$

$$n^2 - 1 = \frac{n(n-1)}{2}$$

$$2n^2 - 2 = n^2 - n$$

$$n^2 + n - 2 = 0$$

$$(n-1)(n+2) = 0$$

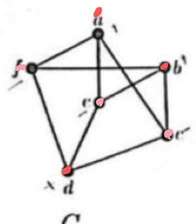
$n = 1$ → $n < 2$

$n = -2$ → impossible

So there exists no complete-- —

(b) (i) Determine whether the graph G below is bipartite. If so, find a bipartite representation. [2]

(ii) Represent G with an adjacency matrix. [1]



i) color theorem

yes its bipartite, and we can find the sets of vertices to represent it by

$V_1 = \{a, b, d\}$

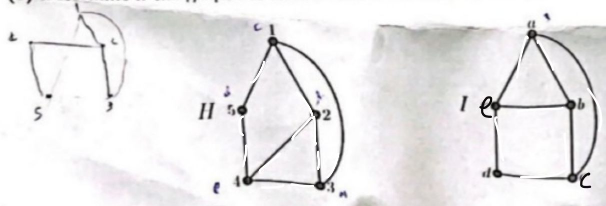
$V_2 = \{f, e, c\}$, where $V_1 \cap V_2 = \emptyset$

$V_1 \cup V_2 = V$ {set of vertices

ii)

	a	b	c	d	e	f
a	0	0	1	0	1	1
b	0	0	1	0	1	1
c	1	1	0	1	0	0
d	0	0	1	0	1	1
e	1	1	0	1	0	0
f	1	1	0	1	0	0

(c) Determine if the graphs H and I below are isomorphic. [2]



1

H:

|V| = 5

|E| = 7

deg(1) = 3

deg(2) = 3

deg(3) = 3

deg(4) = 3

deg(5) = 2

I:

|V| 5

|E| = 7

deg(a) = 3

deg(b) = 3

deg(c) = 3

deg(d) = 2

deg(e) = 3

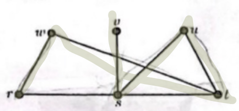
H	1	2	3	4	5
I	a	b	c	e	d

yes, they're isomorphic

Q4. (a) For the graph J below, find a spanning tree with root r,

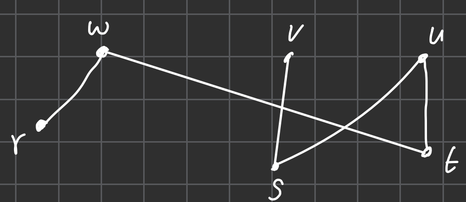
(i) using *depth-first* search; [1]

(ii) using *breadth-first* search. [1]



J

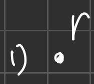
i) longest path from root r : r w t u s v



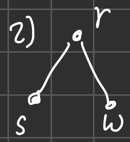
don't add the edges cause they'll result in a circuit

← spanning tree

ii)

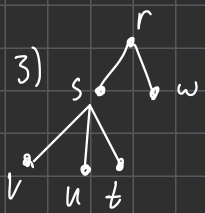


root



level one vertices

3)



level two vertices

8

(b) If T is a tree with degree-sequence $4, 3, 2, x, x, x, x, x$, then find the value of x. [1]

$|V| = 8$ in a tree, $|E| = n - 1$, where $n = |V|$

$E = 8 - 1 = 7$

then hand-shaking theorem: $2e = \sum \deg(v)$

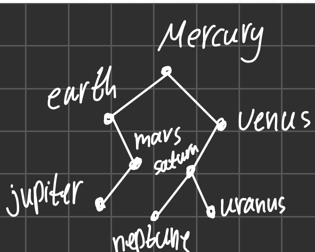
$2 \times 7 = 4 + 3 + 2 + x + x + x + x + x$

$14 = 9 + 5x$

$5 = 5x$

$x = 1$

(c) Using alphabetical order, form a binary search tree for the words: *Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune.* [2]



Q5. **(a)** Let $f(x,y,z) = x\bar{y} + z$ be a Boolean function.

(i) Find the complete sum-of-products expansion (CSP) of f. [2]

(ii) Find the complete product-of-sums expansion (CPS) of f. [2]

i) CSP(f):

$$f = x\bar{y}1 + 11z$$

$$= x\bar{y}(z+\bar{z}) + (x+\bar{x})(y+\bar{y})z \quad \text{unit property}$$

$$= x\bar{y}z + x\bar{y}\bar{z} + xyz + x\bar{y}z + \bar{x}yz + \bar{x}\bar{y}z \quad \text{distribution}$$

$$CSP(f) = x\bar{y}z + x\bar{y}\bar{z} + xyz + \bar{x}yz + \bar{x}\bar{y}z$$

ii) $CPS(f) = [CSP(f^d)]^d$

1) $f^d = (x+\bar{y})z$

2) $CSP(f^d) = xz + \bar{y}z$

$$= x1z + 1\bar{y}z$$

$$= x(y+\bar{y})z + (x+\bar{x})\bar{y}z \quad \text{unit property}$$

$$= xyz + x\bar{y}z + x\bar{y}z + \bar{x}\bar{y}z$$

$$CSP(f^d) = xyz + x\bar{y}z + \bar{x}\bar{y}z$$

3) $[CSP(f^d)]^d = (x+y+z)(x+\bar{y}+z)(\bar{x}+\bar{y}+z)$

So $CPS(f) = (x+y+z)(x+\bar{y}+z)(\bar{x}+\bar{y}+z)$

(b) Let $g(x,y,z) = xyz + xy\bar{z} + x\bar{y}\bar{z} + x\bar{y}z + \bar{x}yz + \bar{x}y\bar{z} + \bar{x}\bar{y}z$ be a Boolean function.
(i) Build the Karnaugh map of g. [1]
(ii) Simplify g (i.e., write it in MSP form). [2]

i)

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	1	1	1	1
$\bar{x}$	1	1		1

ii) $x + z + y$

Final 44 S2

Q1. (a) Without using truth tables show that $[(p \to q) \wedge p] \to q$ is a tautology. (3)

$\equiv [(\neg p \vee q) \wedge p] \to q$ — as $p \to q \equiv \neg p \vee q$

$\equiv [(p \wedge \neg p) \vee (p \wedge q)] \to q$ — distributive law

$\equiv [F \vee (p \wedge q)] \to q$ — negation law

$\equiv (p \wedge q) \to q$ — identity law

$\equiv \neg(p \wedge q) \vee q$ — as $p \to q \equiv \neg p \vee q$

$\equiv \neg p \vee \neg q \vee q$ — de morgan's

$\equiv \neg p \vee T$ — negation law

$\equiv T$ — domination law

tautology