King Saud University
Department of Mathematics

Final Exam in Math 151, T2-1444H.

Calculators are definitely not allowed

(The exam is 2-pages long)

Q1. (a) Without using truth tables show that $[(p \to q) \wedge p] \to q$ is a tautology. (3)

(b) Use induction to show that $4 + 12 + 20 + \ldots + (8n - 4) = (2n)^2$ for all $n \geq 1$. (4)

(c) Suppose m and n are integers. Use a proof by contraposition to show that:
if $4 \mid (m^2 + n^2)$, then m or n is even. (3)

Q2. (a) Let E be the relation on $\mathbb{R} - \{0\}$ defined as follows:

$$xEy \text{ if and only if } \frac{x}{y} \text{ is a } \textbf{positive} \text{ rational number.}$$

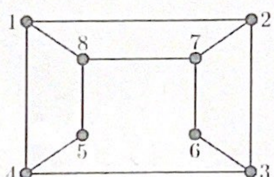
(i) Show that E is an equivalence relation. (3)
(ii) Find $[1]$. (1)
(iii) Is $\sqrt{27}$ in $[-\sqrt{12}]$? (Justify your answer.) (1)

(b) Let $P = \{(1,1), (2,1), (2,2), (3,1), (3,3), (4,1), (4,4), (5,1), (5,5)\}$ be a relation on $A = \{1, 2, 3, 4, 5\}$.

(i) Show that P is a partial order. (3)
(ii) Is P a total order? (Justify your answer.) (1)
(iii) Represent P by a Hasse diagram. (1)

Q3. (a) Find the number of vertices of the complete bipartite graph $K_{n,2n^2}$ which has 16 edges. (2)

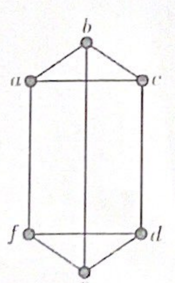
(b) Determine whether the following graph G is bipartite. If so, find a bipartite representation. (2)



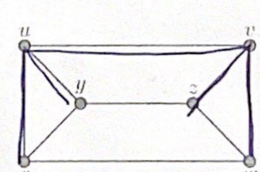
G

(c) Determine if the graphs H and I below are isomorphic. (2)

H



I



Q4. **(a)** For the graph I in **Q3(c)**, find a spanning tree with root v,
(i) using *depth-first* search; (1)
(ii) using *breadth-first* search. (1)

(b) Show that a tree with 12 vertices cannot be a complete graph. (1)

(c) Using alphabetical order, form a binary search tree for the words:
January, February, March, April, May, June, July, August. (2)

A B C D E F G H
I J K L M N O P Q
R S T U V W X Y Z

Q5. **(a)** Without using tables, prove the following Boolean identity. (2)

$$\overline{x\overline{y}+y} = \overline{x+y}\,.$$

(b) Let $f(x,y,z) = (x+yz)(y+\overline{z})$ be a Boolean function.
(i) Find the complete sum-of-products expansion (CSP) of f. (2)
(ii) Find the complete product-of-sums expansion (CPS) of f. (2)

(c) Let $g(x,y,z) = xy\overline{z} + x\overline{y}\,\overline{z} + \overline{x}yz + \overline{x}y\overline{z} + \overline{x}\,\overline{y}z$ be a Boolean function.
(i) Build the Karnaugh map of g. (1)
(ii) Simplify g (i.e., write it in MSP form). (2)

Final Exam: 2^{nd} Sem 1444 ①

Malek Z.A.

Q.(a) $[(p\to q)\wedge p]\to q \overset{?}{\equiv} T$

$[(p\to q)\wedge p]\to q \equiv \neg[(p\to q)\wedge p]\vee q$

$\equiv \neg(p\to q)\vee\neg p\vee q \equiv \neg(\neg p\vee q)\vee(\neg p\vee q)$

$\equiv \neg x\vee x \equiv T \quad : \quad x:\neg p\vee q$.

(b) Let $p(n)$: "$4+12+20+\dots+(8n-4)=(2n)^2$"

Basis step: $n=1$, $L.H.S = 8(1)-4 = 8-4 = \boxed{4}$
$R.H.S = (2(1))^2 = 2^2 = \boxed{4}$
$L.H.S = R.H.S$ $\Downarrow$ $\therefore p(1)$ is true

Inductive step: Let $K\geq 1$, and assume that $p(K)$ is true. $4+12+20+\dots+(8K-4)=(2K)^2$ $*$

will prove $p(K+1)$ is true.

[our goal: $4+12+20+(8K-4)+(8K+4) = [2(K+1)]^2 = 4(K+1)^2$]?

from(*) $\xRightarrow{+(8K+4)}$ $4+12+20+(8K-4)+(8K+4) = (2K)^2+(8K+4)$
$= 4K^2+8K+4$
$= 4(K^2+2K+1)$
$= 4(K+1)^2$

$\Rightarrow \therefore p(K+1)$ is true $\Rightarrow$ So $p(n)$ is true for all $n\geq 1$

#

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(c) $4 \mid (m^2+n^2) \longrightarrow m \vee n$ is even.

$p \longrightarrow q$

Assume $\neg q \equiv$ m and n are <u>odd</u>

$\Rightarrow \begin{array}{l} m = 2K+1 \\ n = 2h+1 \end{array} : K, h \in \mathbb{Z} \begin{array}{l} \Rightarrow m^2 = 4K^2+4K+1 \\ \Rightarrow n^2 = 4h^2+4h+1 \end{array}$

$(+)$

$m^2+n^2 = 4K^2+4K+4h^2+4h+2$

$= 4(\underbrace{K^2+K+h^2+h}_{M})+2$

$m^2+n^2 = 4M+2 \quad : K^2+K+h^2+h = M \in \mathbb{Z}$

$\Rightarrow \therefore 4 \nmid (m^2+n^2) \equiv \neg p$

$\therefore \neg q \longrightarrow \neg p \Rightarrow \therefore p \longrightarrow q$ is true.

Q2 (a) Let E be the relation on $\mathbb{R}-\{0\}$, defined as

$xEy \iff \frac{x}{y}$ is a positive rational number

$\left(\therefore \frac{x}{y} = m \in \mathbb{Q}^{+}\right)$

(i) Show that E is an equivalence relation

① $\forall x \in \mathbb{R}-\{0\}$, $\frac{x}{x} = 1 \in \mathbb{Q}^{+} \Rightarrow \therefore xEx \Rightarrow E$ is refl.

② $x, y \in \mathbb{R}-\{0\}$: $xEy \Rightarrow \frac{x}{y} \in \mathbb{Q}^{+}$

$\therefore E$ is symm. $\Rightarrow \frac{y}{x} \in \mathbb{Q}^{+} \Rightarrow \therefore yEx$

③ $x, y, z \in \mathbb{R}-\{0\}$: $xEy \Rightarrow \frac{x}{y} = m_1 \in \mathbb{Q}^{+}$

$\wedge$

$yEz \Rightarrow \frac{y}{z} = m_2 \in \mathbb{Q}^{+}$

$*$

$\Rightarrow \frac{x}{y} \cdot \frac{y}{z} = \frac{x}{z} = m_1 . m_2 = m \in \mathbb{Q}^{+}$

$\Rightarrow \therefore xEz \Rightarrow \therefore E$ is transitive.

① & ② & ③ $\Rightarrow E$ is an Equiv. Rel. on $\mathbb{R}^{*} = \mathbb{R}-\{0\}$

(ii) Find $[1] = \{x \in \mathbb{R}^{*} : xE1 \Rightarrow \frac{x}{1} = x \in \mathbb{Q}^{+}\} = \mathbb{Q}^{+}$

(iii) is $\sqrt{27} \in [-\sqrt{12}]$? why?

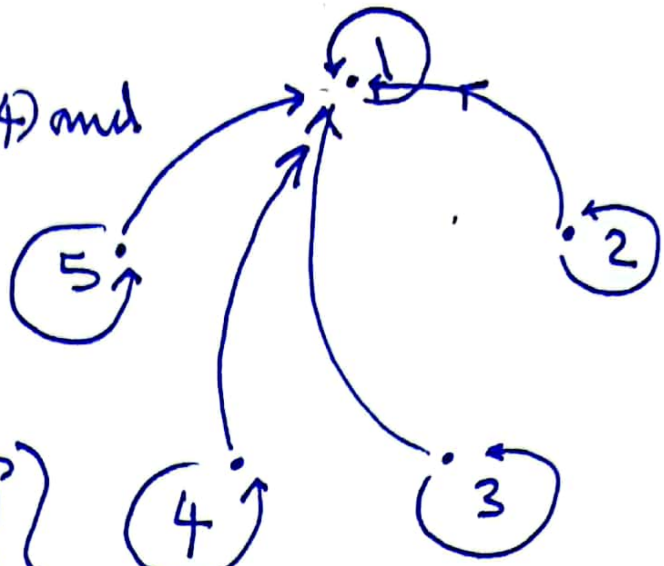
$\because \frac{\sqrt{27}}{-\sqrt{12}} = -\frac{3\cancel{\sqrt{3}}}{2\cancel{\sqrt{3}}} = -\frac{3}{2} \notin \mathbb{Q}^{+} \Rightarrow \therefore \sqrt{27} \not E (-\sqrt{12})$

$\Rightarrow \therefore \sqrt{27} \notin [-\sqrt{12}]$ #

Malek Z. A. (4)

Q_2: (b) $P = \{(1,1),(2,1),(2,2),(3,1),(3,3),(4,1),(4,4),(5,1),$
$(5,5)\}$

(i) ① $\because (1,1),(2,2),(3,3),(4,4)$ and
$(5,5) \in P$
$\Rightarrow \therefore P$ is refl.

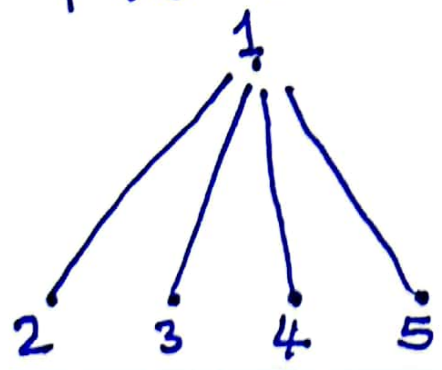


② $\because (2,1) \in P$, but $(1,2) \notin P$
$(3,1) \in P$, but $(1,3) \notin P$
$(4,1) \in P$, but $(1,4) \notin P$
$(5,1) \in P$, but $(1,5) \notin P$
$\therefore P$ is antisymm.

③ $\because (2,1) \wedge (1,1) \in P \Rightarrow (2,1) \in P$ $\Rightarrow P$ is transitive
and same the others

from ①, ② and ③ $\Rightarrow \therefore P$ is a partial order.

(ii) $2,3 \in A$, $\because (2,3) \wedge (3,2) \notin P \Rightarrow \therefore 2,3$ incomparable
$\therefore P$ is not a total order.



- Hasse diagram -

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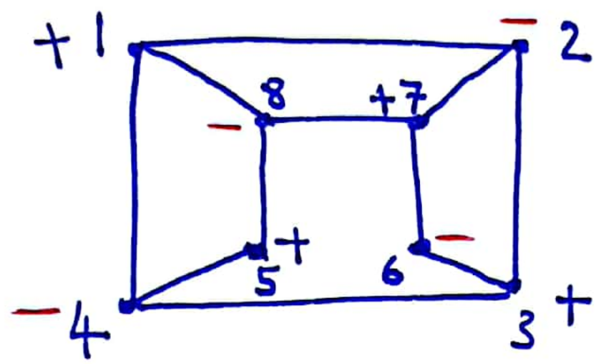
Q_3. (a) $|E(K_{n,2n^2})| = n(2n^2) = 16$

$\Rightarrow 2n^3 = 16 \Rightarrow n^3 = 8 \Rightarrow \therefore \boxed{n=2}$

$|V(K_{n,2n^2})| = n + 2n^2 = 2 + 2(2)^2 = 2+8 = 10$

(b).



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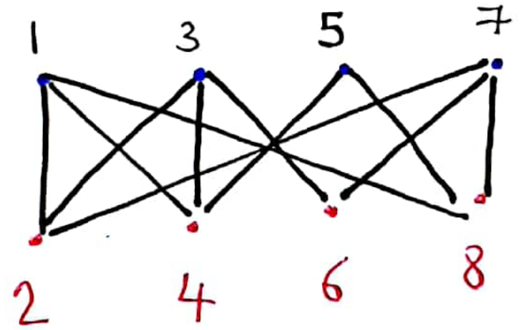

G is bipartite

(c)

x	a	b	c	d	e	f
f(x)	v	z	w	x	y	u

$\therefore H \cong I$

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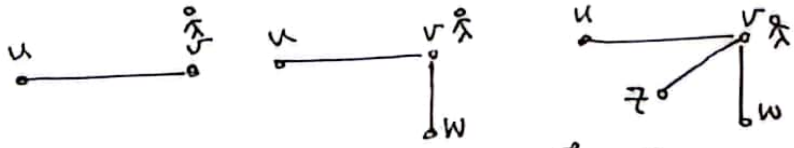
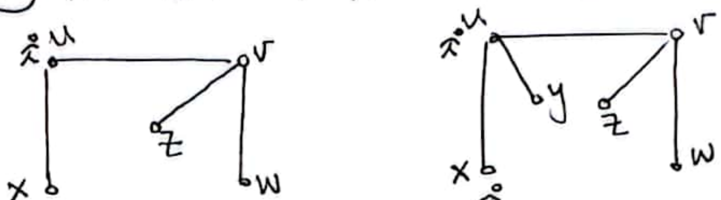
Q. 4

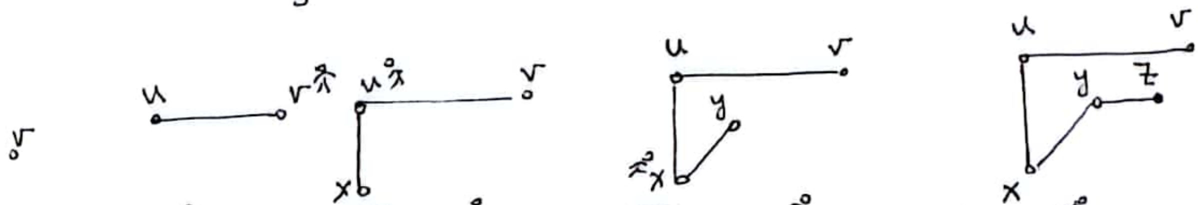


(a) (i) $T_0: v$, $T_1: vu$, $T_2: vuw$, $T_3: vuwz$

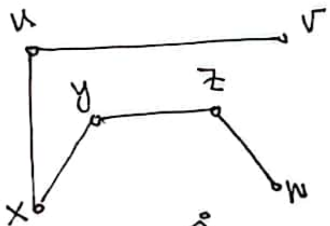


$T_4: vuwzx$, $T_5: vuwzxy$

(ii)



$T_0: v$, $T_1: vu$, $T_2: vux$, $T_3: vuxy$, $T_4: vuxyz$



$T_5: vuxyzw$

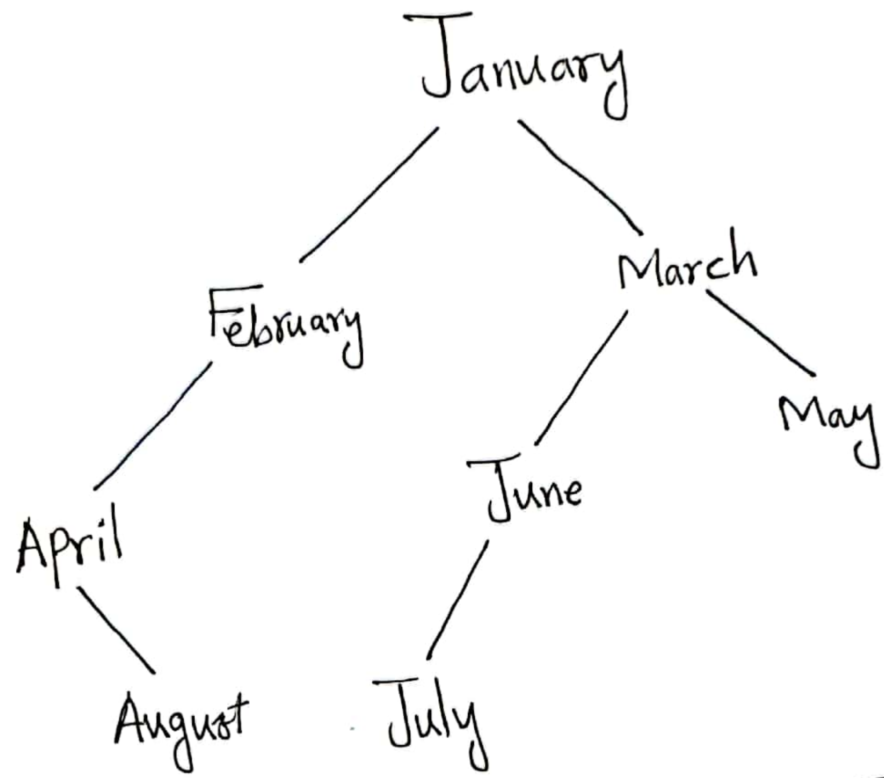
(b) T is a tree with $|V| = 12 \Rightarrow |E| = |V| - 1 = 12 - 1 = \boxed{11}$

but, $|E(K_{12})| = \frac{12(12-1)}{2} = \boxed{66}$.

$\Rightarrow \therefore |E(T)| = 11 \neq |E(K_{12})| = 66 \Rightarrow \therefore T \ncong K_{12}$

Q4.

(c)



Malek Z. A. (7)

Q5 (a) $\overline{x\bar{y}+y} \overset{?}{=} \overline{x+y}$

$$\overline{(x\bar{y}+y)} = \overline{x\bar{y}}\cdot\bar{y} = (\bar{x}+y)\cdot\bar{y} = \bar{x}\,\bar{y} + y\bar{y}$$

$$= \bar{x}\,\bar{y} + 0 = \bar{x}\cdot\bar{y} = \overline{(x+y)} .$$

(b) $f(x,y,z) = (x+yz)(y+\bar{z}) = xy + x\bar{z} + yz$ SP.

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	1	1	1	0
$\bar{x}$	1	0	0	0

$\Rightarrow csp(f) = xyz + xy\bar{z} + x\bar{y}\bar{z} + \bar{x}yz$

$csp(\bar{f}) = \bar{x}y\bar{z} + \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z + x\bar{y}z$

$cps(f) = \left[\overline{csp(\bar{f})}\right] = (x+\bar{y}z)(x+y+z)(x+y+\bar{z})$
$(\bar{x}+y+\bar{z})$.

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(c) $g(x,y,z) = xy\bar{z} + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}y\bar{z} + \bar{x}\bar{y}z$

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x		1	1	
$\bar{x}$	1	1		1

$Msp(g) = x\bar{z} + \bar{x}y + \bar{x}z$