King Saud University
Department of Mathematics

Final Exam in Math151, Semester 2, 1443H.
Calculators are not allowed
(The exam is <u>two-pages</u> long)

Q1. (a) Without using truth tables show that $(p \to q) \to r$ is logically equivalent to $(\neg r \to p) \wedge (q \to r)$. (2pts)

(b) Use induction to show the following for every $n \geq 2$:

$$(\frac{1}{2}-\frac{1}{4})(\frac{1}{2}-\frac{1}{6})(\frac{1}{2}-\frac{1}{8})\cdots(\frac{1}{2}-\frac{1}{2n})=\frac{1}{2^{n-1}n}. \text{ (4pts)}$$

(c) Assuming that $\sqrt{6}$ is irrational, use a proof by way of contradiction to show that $\frac{5\sqrt{6}-3}{2}+4$ is irrational. (2pts)

Q2. (a) Let R be the relation on $\mathbb{Z}$ (the set of integers) defined by mRn if and only if $2 \mid (m+n)$.
(i) Show that R is an equivalence relation. (3pts)
(ii) Find the equivalence classes [0] and [1]. (2pts)

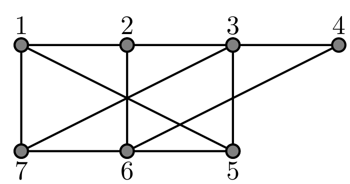
(b) Let P be the relation on $\{1,3,5\}$ defined by: $aPb \iff a < b+2$.
(i) List all ordered pairs of P. (2pts)
(ii) Represent P by a digraph. (1pts)
(iii) Show that P is a partial order. (3pts)
(iv) Is P a total order? (Justify your answer.) (1pts)
(v) Represent P by a Hasse diagram. (1pts)

Q3. (a) Let G be a graph with degree-sequence: $a-3, a-2, a-1, a, a+2$. Find the value of a if G has 8 edges. (2pts)

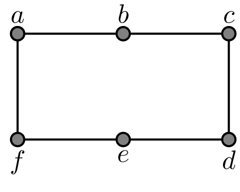
(b) Let T be a tree with 12 edges. Find the number of vertices of T and the number of edges of the complement $\overline{T}$ of T. (2pts)

(c) Determine whether the graph H below is bipartite. If so, give a bipartite representation. (2pts)

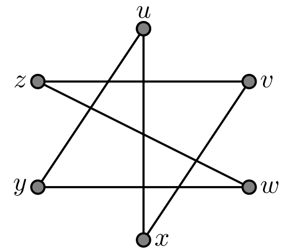
H



(d) Determine whether the following graphs I and J are isomorphic. (2pts)

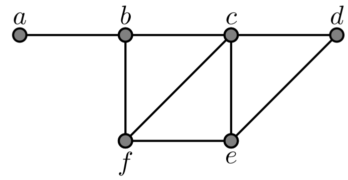


I



J

Q4. **(a)** For the graph L below, find a spanning tree with root b,



L

(i) using *depth-first* search; (1pts)
(ii) using *breadth-first* search. (1pts)

(b) Using alphabetical order, form a binary search tree for the words *purple, black, yellow, green, blue, white, grey.* (2 pts)

Q5. **(a)** Let $f(x, y, z) = \overline{\overline{x}y + \overline{y}z + \overline{y}}$ be a Boolean function.
(i) Find the complete sum-of-products expansion (CSP) of f. (2pts)
(ii) Find the complete product-of-sums expansion (CPS) of f. (2pts)

(b) Let $g(x, y, z, w) = xyzw + xyz\overline{w} + xy\overline{z}w + x\overline{y}\,\overline{z}w + \overline{x}\,\overline{y}zw + \overline{x}\,\overline{y}z\overline{w} + \overline{x}yzw + \overline{x}y\overline{z}\,\overline{w} + \overline{x}y\overline{z}w$ be a Boolean function.
(i) Build the Karnaugh map of g. (1pts)
(ii) Simplify g (i.e., write in MSP form). (2pts)

Q1. (a) Without using truth tables show that $(p \to q) \to r$ is logically equivalent to $(\neg r \to p) \land (q \to r)$. (2pts)

$(P \to q) \to r \overset{?}{\equiv} (-r \to P) \land (q \to r)$

LHS: $-(P \to q) \lor r \equiv$ Because $P \to q \equiv -P \lor q$

$-(-P \lor q) \lor r \equiv$ Because $P \to q \equiv -P \lor q$

$(P \land -q) \lor r \equiv$ Double negation law $-(-P) \equiv P$ AND de morgan law $-(P \lor q) \equiv -P \land -q$

$(P \lor r) \land (-q \lor r) \equiv$ Distributive law $P \lor (q \land r) \equiv (P \lor q) \land (P \lor r)$

$(r \lor P) \land (-q \lor r) \equiv$ commutative law $P \lor q \equiv q \lor P$

$(-r \to P) \land (q \to r) \equiv$ RHS :) Because $-P \lor q \equiv P \to q$

(b) Use induction to show the following for every $n \geq 2$:

$$(\frac{1}{2} - \frac{1}{4})(\frac{1}{2} - \frac{1}{6})(\frac{1}{2} - \frac{1}{8}) \cdots (\frac{1}{2} - \frac{1}{2n}) = \frac{1}{2^{n-1}n}. \text{ (4pts)}$$

1- Basis step: Show that $P(2)$ is true $[n=2]$

LHS: $P(2) = (\frac{1}{2} - \frac{1}{4}) = \frac{1}{4}$

RHS: $P(2) = \frac{1}{2} - \frac{1}{2(2)} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

$\frac{1}{4} = \frac{1}{4}$ ✓

LHS = RHS then $P(2)$ is true.

2- Inductive step:

Assume $P(K)$ is true $[n=K]$ that is:

$$(\frac{1}{2} - \frac{1}{4})(\frac{1}{2} - \frac{1}{6})(\frac{1}{2} - \frac{1}{8}) \cdots (\frac{1}{2} - \frac{1}{2K}) = \frac{1}{2^{K-1} \cdot K}$$ } Inductive Hypotheses

we want to show $P(K+1)$ is true $[n=K+1]$, that is

we want to show

$$\left(\frac{1}{2}-\frac{1}{4}\right)\left(\frac{1}{2}-\frac{1}{6}\right)\left(\frac{1}{2}-\frac{1}{8}\right)\cdots\cdots\left(\frac{1}{2}-\frac{1}{2K+2}\right)=\frac{1}{2^K\cdot(K+1)}$$

LHS:

$$\frac{1}{2^{K-1}\cdot K}\cdot\left(\frac{1}{2}-\frac{1}{2K+2}\right)=$$

$$\frac{1}{2^{K-1}\cdot K}\cdot\left(\frac{1}{2}\cdot\frac{K+1}{K+1}-\frac{1}{2K+2}\right)=$$

$$\frac{1}{2^{K-1}\cdot K}\cdot\left(\frac{K+1-1}{2K+2}\right)=$$

$$\frac{1}{2^{K-1}\cdot\cancel{K}}\cdot\left(\frac{\cancel{K}}{2K+2}\right)=$$

$$\frac{1}{2^{K-1}\cdot(2K+2)}=\frac{1}{\underbrace{2^{K-1}(2K)+2^{K-1}(2)}_{\pm\text{ Add exponent}}}=\frac{1}{2^K\cdot K+2^K}$$

$$\frac{1}{\underbrace{2^K[K+1]}_{2^K\text{ as common factor}}}=RHS$$

$\therefore$ we verfied $P(K+1)$ is true

$\rightarrow$ thus by Mathmatical induction + from BS and IS we proved $P(n)$ is true $\forall n\geq 2$

(c) Use contradiction to show that if n^3+5 is an odd integer, then n is even. (2pts)

P: n^3+5 odd integer   q: n even

$P \to q$: if n^3+5 odd + integer $\longrightarrow$ n even

By Contradiction we Assume P and $\sim q$ are true, that is:

P: n^3+5 odd integer   is true

$\sim q$: n is odd   is true

then:

$\sim q$: $n = 2K+1$, $\exists K \in \mathbb{Z}$

$n^3+5 = (2K+1)^3+5$

$= 8K^3+12K^2+6K+1+5$

$= 8K^3+12K^2+6K+6$

$= 2[4K^3+6K^2+3K+3]$

$= 2m$, $\exists m \in \mathbb{Z}$ s.t $m = 4K^3+6K^2+3K+3$

$\to$ n^3+5 is even

hint: use this formula:
$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$

So we got a contradiction $P \wedge \sim P$ are true

$\therefore$ By contradiction if n^3+5 odd + integer $\longrightarrow$ n even   is True

Q2

(a)

Q2. (a) Let R be the relation on $\mathbb{Z}$ (the set of integers) by mRn if and only if $2 \mid (m+n)$.
(i) Show that R is an equivalence relation. (3pts)
(ii) Find the equivalence classes $[0]$ and $[1]$. (2pts)

i – ① $\forall m \in \mathbb{Z}$ $\quad 2 \mid (m+m)$ $\quad$ that is $\quad 2 \mid 2m \Rightarrow$

$2m = 2k \;,\; \underset{\Rightarrow k=m}{\exists k \in \mathbb{Z}}$ $\quad \therefore \forall m \in \mathbb{Z} \;\; mRm$

$\therefore$ R is Reflixive on $\mathbb{Z}$

② $\forall n \forall m \in \mathbb{Z} \quad mRn \longrightarrow nRm$

$2 \mid (m+n) \Rightarrow m+n = 2j \quad \exists j \in \mathbb{Z}$

$\Rightarrow n+m = 2j$

$\Rightarrow 2 \mid (n+m)$

$\Rightarrow nRm$

$\therefore$ R is symmetric on $\mathbb{Z}$

③ $\forall n \forall m \forall s \in \mathbb{Z} \quad mRn \wedge nRs \longrightarrow mRs$

$2 \mid m+n \;\wedge\; 2 \mid n+s \rightarrow$

$\underbrace{m+n = 2k \;\wedge\; n+s = 2j}_{\text{Add them together}} \Rightarrow \quad \exists k, j \in \mathbb{Z}$

$m+n+n+s = 2k+2j \Rightarrow m+2n+s = 2k+2j \Rightarrow m+s = 2k+2j+2n \Rightarrow$

$m+s = 2[k+j+n] \Rightarrow m+s = 2a \quad , \exists a \in \mathbb{Z} \text{ s.t } a = k+j+n$

↳ $2 \mid m+s \Rightarrow mRs$

$\therefore$ R is transitive on $\mathbb{Z}$

$\therefore$ from 1, 2, 3 R is an equavelance relation

ii – Firstly find $[a]$

$[a] = \{x \in \mathbb{Z} : xRa\} = \{x \in \mathbb{Z} : 2 \mid x+a\} = \{x \in \mathbb{Z} : x+a = 2K, \exists K \in \mathbb{Z}\}$

$= \{x = 2K - a\} \Rightarrow [a] = \{2K - a\}$

$[0] = \{2K - 0\} = \{2K\} \Rightarrow \{\dots -6, -4, -2, 0, 2, 4, 6 \dots\}$

$[1] = \{2K - 1\} \Rightarrow \{\dots -7, -5, -3, 1, 3, 5, 7 \dots\}$

(b)

(b) Let P be the relation on $\{1, 3, 5\}$ defined by: $aPb \iff a < b + 2$.

(i) List all ordered pairs of P. (2pts)

(ii) Represent P by a digraph. (1pts)

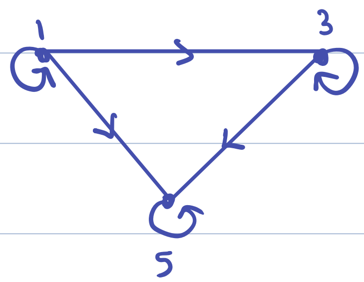
(iii) Show that P is a partial order. (3pts)

(iv) Is P a total order? (Justify your answer.) (1pts)

(v) Represent P by a Hasse diagram. (1pts)

i– $P = \{(1,5), (1,3), (3,5), (1,1), (3,3), (5,5)\}$

ii–



suppose $A = \{1,3,5\}$

iii- ① $\forall a \forall b \in A$ aPa $\underset{-a}{a} < \underset{-a}{a+2} \Rightarrow 0<2$

∴ P is Reflixive on A

② $\forall a \forall b \in A$ $aPb \wedge bPa \longrightarrow a=b$

$(1,5) \in P$ but $(5,1) \notin P$ and $(1,3] \in P$ but $(3,1) \notin P$ and

$(3,5) \in P$ but $(5,3) \notin P$

∴ P is antisymmetric on A

③ $\forall a \forall b \forall c \in A$ $aPb \wedge bPc \longrightarrow aPc$

$(1,3) \in P \wedge (3,5) \in P \longrightarrow (1,5) \in P$

∴ P is transitive on A

As P is ref, anti, trans then P is a Partial order

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iv- yes, since for every two elements in A

aPb <u>or</u> bPa [each two elements are comparable]

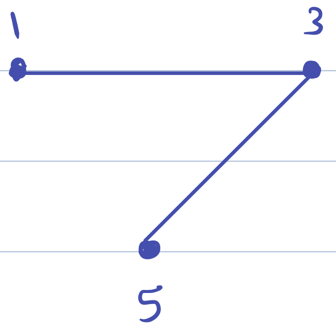
① $(1,3) \in P$

② $(3,5) \in P$

③ $(1,5) \in P$

∴ totaly ordered and $(P, \leq)$ is a poset

$r-$



Q3 ⓐ

Q3. (a) Let G be a graph with degree-sequence: $a-3, a-2, a-1, a, a+2$. Find the value of a if G has 8 edges. (2pts)

$E = \frac{n \cdot r}{2}$

$2m = \sum deg(n)$ where m is num of edges |E| and n is num of vertices |V| ,

$2(8) = (a-3) + (a-2) + (a-1) + a + (a+2) =$

$16 = 5a - 4 \quad \Rightarrow \quad \frac{20}{5} = \frac{5a}{5} \quad \Rightarrow \quad \boxed{a = 4}$

$+4 \qquad +4$

ⓑ

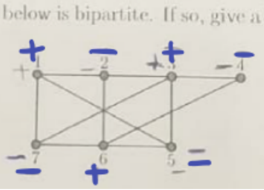
(b) If G is a 5-regular graph with 15 edges, then find the number of vertices of G.

$|E| = \frac{r \cdot n}{2} \quad \Rightarrow \quad 15 = \frac{n \cdot 5}{2} \quad \Rightarrow \quad 30 = 5n \quad \Rightarrow \quad n = 6$

Number of vertices of G: $|V| = 6$

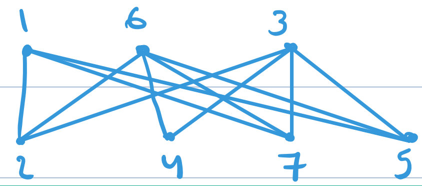
(c) Determine whether the graph H below is bipartite. If so, give a bipartite representation. (2pts)

H

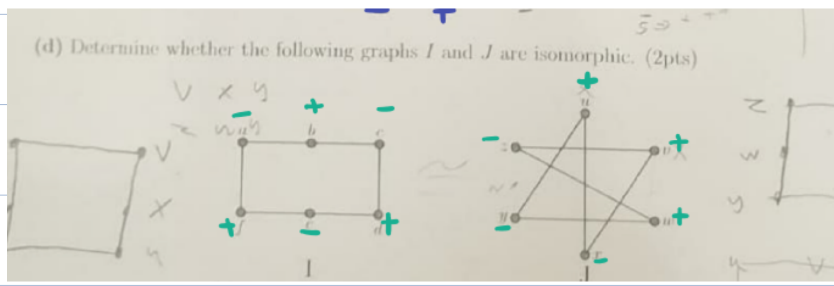


ⓒ $v_1 = \{1, 6, 3\}$ $v_2 = \{2, 4, 7, 5\}$

yes Bipartite as the vertices can be divided into two groups where no two vertices in the same group are directly related



ⓓ

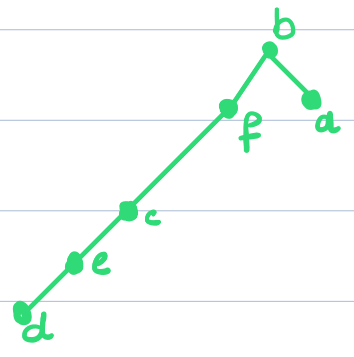


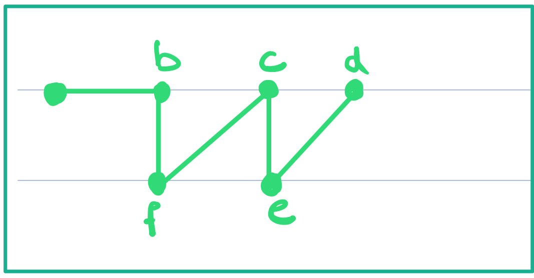
I	a	b	c	d	e	f
J	u	x	v	z	w	y

∴ I and J are Isomorphic

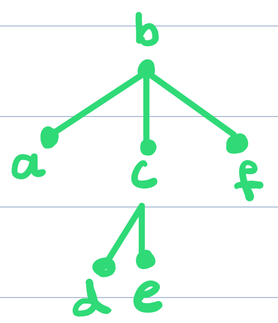
Q4 ⓐ i-

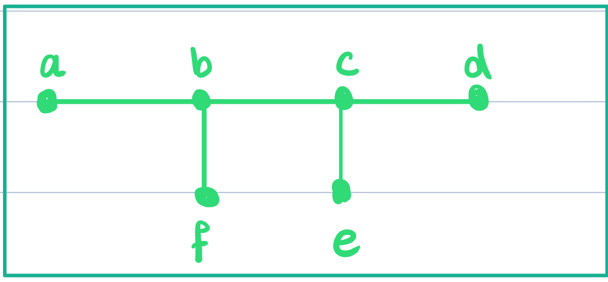
taking longest path b, f, c, e, d



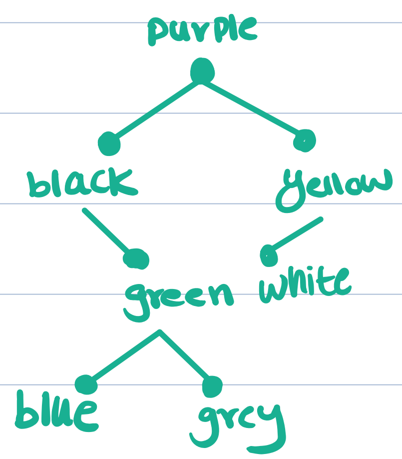


ii -





b



Q5: a

5. (a) Let $f(x, y, z) = \overline{\overline{x}y + \overline{y}z + \overline{y}}$ be a Boolean function.
(i) Find the complete sum-of-products expansion (CSP) of f. (2pts)
(ii) Find the complete product-of-sums expansion (CPS) of f. (2pts)

i- CSP(f) =

$(x+\bar{y})(y+\bar{z})(y) = [xy + x\bar{z} + \bar{y}y + \bar{y}\bar{z}](y) =$

$xyy + x\bar{z}y + \underbrace{\bar{y}yy + \bar{y}\bar{z}y}_{=0} = xy + x\bar{z}y$

$\to xy(z+\bar{z}) + x\bar{z}y = xyz + xy\bar{z} + x\bar{z}y =$

$CSP(f) = xyz + xy\bar{z}$

ii- $CPS(f) = [CSP(f)^d]^d$

① $(f)^d = x\bar{y} + y\bar{z} + y$

② CSP(f) = $x\bar{y}(z+\bar{z}) + y\bar{z}(x+\bar{x}) + y(\bar{x}+x)(z+\bar{z})$

$= x\bar{y}z + x\bar{y}\bar{z} + xy\bar{z} + \bar{x}y\bar{z} + [\bar{x}y + xy](z+\bar{z})$

$= \underline{x\bar{y}z} + \underline{x\bar{y}\bar{z}} + \cancel{xy\bar{z}} + \underline{\bar{x}y\bar{z}} + \underline{\bar{x}yz} + \cancel{\bar{x}y\bar{z}} + \underline{xyz} + \underline{xy\bar{z}}$

$= xyz + x\bar{y}z + \bar{x}y\bar{z} + \bar{x}yz + xy\bar{z} + x\bar{y}\bar{z}$

CPS(f): $= (x+y+z)(x+\bar{y}+z)(\bar{x}+y+\bar{z})(\bar{x}+y+z)(x+y+\bar{z})(x+\bar{y}+\bar{z})$

ⓑ

(b) Let $g(x,y,z) = xyz + xy\bar{z} + x\bar{y}\,\bar{z} + x\bar{y}\,z + \bar{x}\,yz + \bar{x}\,\bar{y}\,\bar{z} + \bar{x}\,\bar{y}z$ be a Boolean function.
(i) Build the Karnaugh map of g. (1pts)
(ii) Simplify g (i.e., write in MSP form). (2pts)

i-

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	1	1	1	1
$\bar{x}$	1		1	1

ii- MSP: $z + x + \bar{y}$

Good luck

Swt.