King Saud University
Department of Mathematics

Final Exam in Math 151
Semester 1, 1443 H.

Calculators are not allowed

The Examination contains 2 pages

Q1. (a) Without using truth tables show that $(p \wedge q) \vee (\neg p \wedge q) \vee (p \wedge \neg q) \equiv p \vee q$. (2pts)

(b) Suppose a, b and c are integers. Prove that if a divides $(b-1)$ and a divides $(c-1)$; then a divides $(bc-1)$. (3pts)

(c) Use induction to show that $1+5+9+\ldots+(4n-3) = n(2n-1)$ for all $n \geq 1$. (4pts)

Q2. (a) Let R be the relation on $\mathbb{Q}$ (the set of rational numbers) defined by xRy if and only if $x-y$ is an integer.

(i) Show that R is an equivalence relation. (3pts)

(ii) Find the equivalence class $[0]$. (1pts)

(iii) Show that $[7/16] \neq [-1/8]$. (1pts)

(b) Let $P = \{(a,a),(a,c),(b,b),(b,d),(c,c),(d,d)\}$ be a relation on the set $A = \{a,b,c,d\}$.

(i) Represent P by a digraph. (1pts)

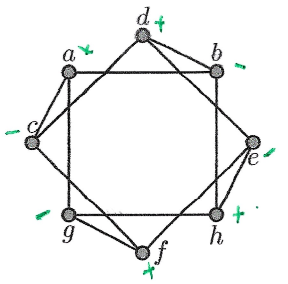
(ii) Show that P is a partial order. (3pts)

(iii) Is P a total order? (Justify your answer.) (1pts)

(iv) Represent P by a Hasse diagram. (1pts)

Q3. (a) Find the number of vertices n of a complete graph K_n which has 36 edges. (2pts)

(b) Let G be the graph below.



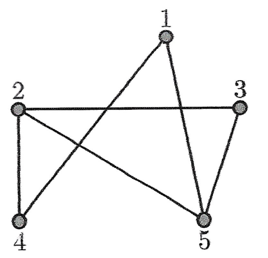
G

(i) Determine whether G is bipartite. (Justify your answer)(2pts)

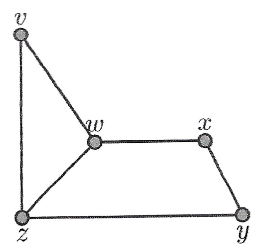
(ii) Find a simple path from a to h of length 4. (1pts)

(iii) Find a cycle from a to a of length 6. (1pts)

(c) Determine whether the following graphs M and N are isomorphic. (Justify your answer)(2pts)

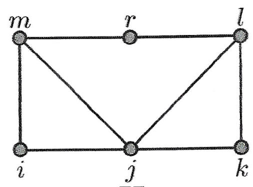


M



N

Q4. (a) For the graph H below, find a spanning tree with root r,



H

(i) using *depth-first* search; (1 pts)

(ii) using *breadth-first* search. (1 pts)

(b) Form a binary search tree for the words *english, chinese, russian, german, arabic, spanish, french.* (using alphabetical order). (2 pts)

Q5. (a) Prove the following Boolean identity. (2pts)

$$(x+y)(x+\overline{y})(\overline{x}+y) = xy.$$

(b) Find the complete sum-of-products expansion (CSP) for the function $f(x, y, z) = (x + \overline{y}z)(\overline{x} + z)$. (2pts)

(c) Find the complete product-of-sums expansion (CPS) for the function $g(x, y, z) = \overline{y} + x\overline{z}$. (2pts)

(d) Use a K-map to simplify the following sum-of-products expansion (i.e., write in MSP form).(2pts)

$$xyz + x\overline{y}\,\overline{z} + x\overline{y}z + \overline{x}y\overline{z} + \overline{x}\,\overline{y}z.$$

Answer final examination
Math 151.
Semester 1 1443.

Question 1: (9 marks)

a) 2

$(p \wedge q) \vee (\neg p \wedge q) \vee (p \wedge \neg q)$

$\equiv (q \wedge (p \vee \neg p)) \vee (p \wedge \neg q)$

$\equiv (q \wedge T) \vee (p \wedge \neg q)$

$\equiv (q \vee p) \wedge (q \vee \neg q) \equiv (q \vee p) \wedge T \equiv q \vee p.$

b) 3

$a \mid b-1 \Rightarrow b-1 = q_1 a. \Rightarrow b = q_1 a + 1.$

$a \mid c-1 \Rightarrow c-1 = q_2 a. \Rightarrow c = q_2 a + 1.$

$bc = (q_1 a + 1)(q_2 a + 1)$

$= q_1 q_2 a^2 + q_1 a + q_2 a + 1$

$\Rightarrow bc - 1 = q_1 q_2 a^2 + q_1 a + q_2 a$

$= a(q_1 q_2 a + q_1 + q_2)$

$\Rightarrow a \mid bc - 1.$

c) 4

$P(n): 1+5+9+\dots+(4n-3) = n(2n-1) \quad \forall n \geq 1.$

B.S: $P(1): 1 = 1(2-1) = 1$ True.

I.S. Let $k \geq 1$, we assume that $P(k)$ is true

$1+5+9+\dots+(4k-3) = k(2k-1)$ I.H.

and we prove that $P(k+1)$ is true.

$1+5+9+\dots+(4(k+1)-3) = (k+1)(2(k+1)-1)$
$= (k+1)(2k+1).$

$1+5+9+\dots+(4k+1)$

$= 1+5+9+\dots+(4k-3)+(4k+1)$

$= k(2k-1)+4k+1.$

$= 2k^2 - k + 4k + 1$

$= 2k^2 + 3k + 1 = L.H.S.$

R.HS: $(k+1)(2k+1) = 2k^2 + 2k + k + 1$
$= 2k^2 + 3k + 1 = L.H.S$

So $P(k+1)$ is true

$\left.\begin{array}{l} \Rightarrow P(1) \text{ true} \\ P(k) \rightarrow P(k+1) \ \forall k \geq 1 \end{array}\right\} \Rightarrow \forall n \geq 1 \ P(n)$ is true.

Question 2: (12 marks)

a)

(i) 3 • Let $x \in \mathbb{Q}$; $x - x = 0 \in \mathbb{Z}$

$\Rightarrow x R x \Rightarrow R$ is reflexive (1).

• Let $x, y \in \mathbb{Q}$ such that xRy

$\Rightarrow x - y \in \mathbb{Z} \Rightarrow y - x = -(x-y) \in \mathbb{Z}$

$\Rightarrow yRx$

$\Rightarrow R$ is symmetric (2).

• Let $x, y, z \in \mathbb{Q}$ such that xRy and yRz

$\Rightarrow x - y \in \mathbb{Z}$ and $y - z \in \mathbb{Z}$

$\Rightarrow (x - y + y - z) \in \mathbb{Z}$

$\Rightarrow x - z \in \mathbb{Z} \Rightarrow xRz$

$\Rightarrow R$ is transitive (3)

(1), (2) and (3) $\Rightarrow R$ is an equivalence relation on $\mathbb{Z}$.

(ii) 1

$[0] = \{x \in \mathbb{Q};\ xR0\}.$

$= \{x \in \mathbb{Q};\ x - 0 \in \mathbb{Z}\}$

$= \{x \in \mathbb{Q};\ x \in \mathbb{Z}\} = \mathbb{Z}.$

(iii) 1

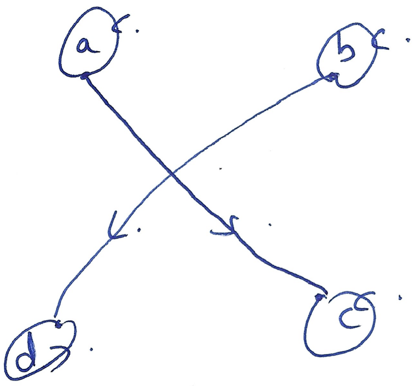
$\frac{7}{16} - (-\frac{1}{8}) = \frac{7}{16} + \frac{1}{8}$

$= \frac{7+2}{16} = \frac{9}{16} \notin \mathbb{Z}.$

$\Rightarrow \frac{7}{16} \not R (-\frac{1}{8}) \Rightarrow [\frac{7}{16}] \neq [-\frac{1}{8}].$

b)

(i) 1



(ii) : 3

- $(a,a),(b,b),(c,c),(d,d) \in P$

$\Rightarrow$ P is reflexive (1)

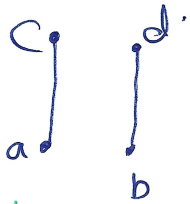
- $(a,c) \in P$; $(c,a) \notin P$
$(b,d) \in P$; $(d,b) \notin P$ $\Big\} \Rightarrow$ P is anti-sym (2)

- P is transitive (3)

(1), (2) and (3) $\Rightarrow$ P is a partial order.

(iii) 1 P is not a total order because a, b are not comparable

(iv) 1



Question 3: (8 marks)

a) 2

$$|E(K_n)| = \frac{n(n-1)}{2} \Rightarrow \frac{n(n-1)}{2} = 36.$$

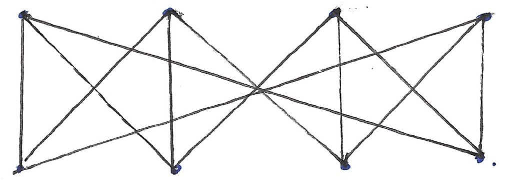
$\Rightarrow n(n-1) = 72.$

so $n = 9$.

K_n contain 9 vertices.

b)

(i) (2pts)



(ii) 1

a, g, f, e, h. — a, c, f, e, h

a, g, f, e, d, c, a

(iii) 1

a, b, d, e, f, c, a.

c) 2

S_1

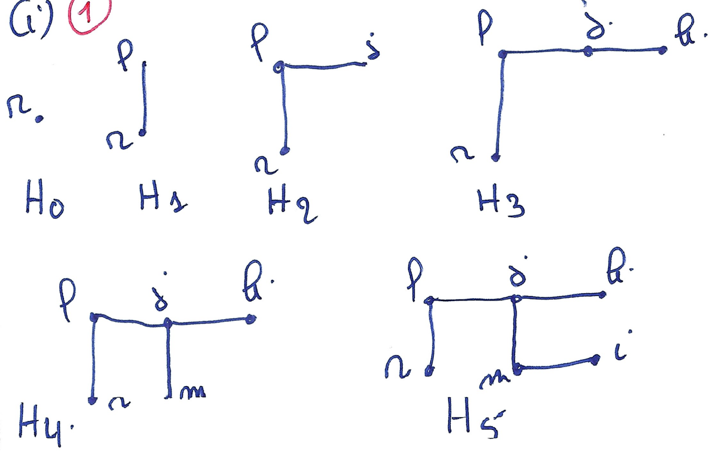
a	2	5	3	4	1
f(a)	w	z	v	x	y

S_2

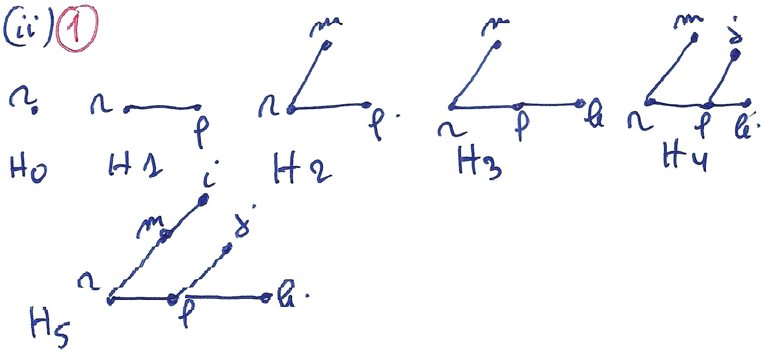
a	2	5	3	4	1
f(a)	z	w	v	y	x

Question 4: (4 marks)

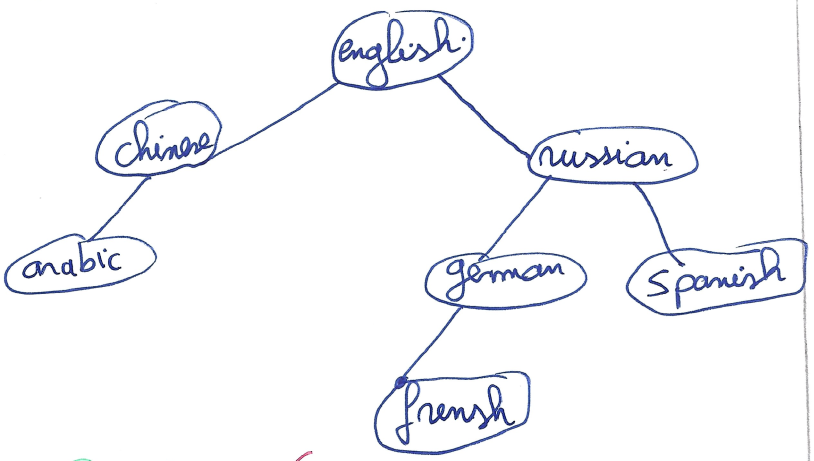
(i) 1



(ii) 1



b) 2



Question 5: (8 marks)

a) 2 $(x+y)(x+\bar{y})(\bar{x}+y)$

$= (x + x\bar{y} + xy + y\bar{y})(\bar{x}+y)$

$= (x + x\bar{y} + xy)(\bar{x}+y)$

$= \cancel{x\bar{x}} + xy + \cancel{x\bar{y}\bar{x}} + \cancel{x\bar{y}y} + \cancel{xy\bar{x}} + xyy.$

$= xy + xy = xy.$

b) 2

$f(x,y,z) = (x+\bar{y}z)(\bar{x}+z)$

$= x\bar{x} + xz + \bar{x}\bar{y}z + \bar{y}zz$

$= xz + \bar{x}\bar{y}z + \bar{y}z$

$= xyz + x\bar{y}z + \bar{x}\bar{y}z + x\bar{y}z + \bar{x}\bar{y}z$

$CSP(f) = xyz + x\bar{y}z + \bar{x}\bar{y}z + x\bar{y}z.$

c) 2

$g(x,y,z) = \bar{y} + x\bar{z}$

$\Rightarrow \bar{g}(x,y,z) = y \cdot (\bar{x}+z)$

$= y\bar{x} + yz.$

$= \bar{x}yz + \bar{x}y\bar{z} + xyz + \bar{x}yz.$

$CSP(\bar{g}) = \bar{x}yz + \bar{x}y\bar{z} + xyz.$

$\Rightarrow CPS(g) = (x+\bar{y}+\bar{z})(x+\bar{y}+z)(\bar{x}+\bar{y}+\bar{z})$

d)

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	1	0	1	1
$\bar{x}$	0	1	0	1

$MSP(h) = xz + x\bar{y} + \bar{y}z + \bar{x}y\bar{z}.$