King Saud University	College of Sciences		Department of Mathematics
Final Examination	Math 132	Semester I (1446)	Time:3 Hours

Question 1: (8 marks)

1. Without using truth tables, prove that the following conditional statement is a Tautology:

$$[(p \to q) \wedge p] \vee (q \to \neg p). \quad \text{(3 marks)}$$

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$L.H.S = [(p \to q) \wedge p] \vee (q \to \neg p)$

$\equiv [(\sim p \vee q) \wedge p] \vee (\sim q \vee \sim p)$ — Table 7 conditional rule

$\equiv [(\sim p \wedge p) \vee (q \wedge p)] \vee (\sim q \vee \sim p)$ — distiribution low

$\equiv [F \vee (q \wedge p)] \vee \sim(q \wedge p)$ — Demo-gans low + Table 6

$\equiv (q \wedge p) \vee \sim(q \wedge p)$ — Identity low

$\equiv T$ — by $q \wedge \sim p \equiv T$.

2. Let $n \in \mathbb{N}$. Show that: if n^2 is odd, then $1 - n$ is even. (2 marks)

by contra position $\neg q \to \neg p$

$1-n$ is odd $\overset{??}{\Longrightarrow}$ n^2 is even

suppose $1-n$ is odd

$\Rightarrow 1-n = 2k+1$ for some $k \in \mathbb{Z}$ 0.25 0.25

$\Rightarrow n = -2k$ 0.25

$\Rightarrow n^2 = (-2k)^2 = 4k^2 = 2(2k^2)$ 0.5 $\Rightarrow n^2$ is even 0.25

since $(2k^2) \in \mathbb{Z}$ 0.25

Hence, If n^2 is odd $\Longrightarrow$ $1-n$ is even 0.25

$4n$

3. Use mathematical induction to prove the following statement:

$$9+13+17+\cdots+(4n+5) = n(2n+7), \quad \text{for each integer } n, \text{ with } n \geq 1. \quad \text{(3 marks)}$$

let $P(n)$ be $9+13+\cdots+(4n+5) = n(2n+7)$ $\forall\ n \geq 1$ 0.25

Basis step $P(1)$ is true

$L.H.S = 9 = 1(2(1)+7) = R.H.S$ 0.5

Induction step:

suppose $P(K)$ is true for some $K \geq 1$ 0.5

That is $9+13+\cdots+(4k+5) = k(2k+7)$ → I.H

Then for $k+1$

$L.H.S = 9+13+\cdots+4K+5+4(K+1)+5$

$= K(2K+7)+4(K+1)+5$ 0.5 from I.H. 0.25

$= 2K^2+7K+4K+4+5$

$= 2K^2+11K+9$

$= (K+1)(2K+9)$ 0.5

$= (K+1)(2(K+1)+7)$

$= R.H.S$

Hence $P(K+1)$ is true 0.25

Hence $P(n)$ is true $\forall n \geq 1$ 0.25

Question 2: (15 marks)

1. Let R be the relation on the set $\mathbb{Z}$ defined by

$$a, b \in \mathbb{Z};\ aRb \iff b = -a$$

Decide whether the relation R is reflexive, symmetric, antisymmetric or transitive. Justify your answers. (4 marks)

D Reflexive, since $1 \in \mathbb{Z}$ and $1 \neq -1$
then R is not reflexive.

symetric suppose $a R b$ $\Rightarrow b R a$
$\iff b = -a \iff a = -b$

then R is symetric.

antisymetric, since $1 \neq -1$ and $1 R -1$ and $-1 R 1$
$\Rightarrow$ R is not antisymetric.

transitive, since $1 R -1$ and $-1 R 1$
but $1 \not R 1$, then R is not antisymetric

2. Let E be the relation on the set $\mathbb{Z} - \{0\}$ defined by

$$m, n \in \mathbb{Z} - \{0\};\ mEn \iff 3mn > 0$$

(a) Show that E is an equivalence relation. (3 marks)

(b) Find [1] and [−1]. (2 marks)

a) E is reflexive,

$\forall m \in \mathbb{Z}-\{0\}$ $\quad 3mm = 3m^2 > 0 \Rightarrow mEm$

E is symetric because

$\forall m,n \in \mathbb{Z}-\{0\}$ if $mEn \Rightarrow 3mn > 0$

$\Rightarrow 3nm > 0 \Rightarrow nEm$

E is transitive, because, If $m,n,r \in \mathbb{Z}-\{0\}$

and $mEn \wedge nEr \Rightarrow 3mn > 0 \wedge 3nr > 0$

$\Rightarrow 3^2mn^2r = 3n^2(3mr) > 0$

$\Rightarrow 3mr > 0 \Rightarrow mEr$

From the above E is reflexive, symetric and antisymetric $\Rightarrow E$ is an equivalence relation

b) $[1] = \{m \in \mathbb{Z}-\{0\} : 1Em\}$

$= \{m \in \mathbb{Z}-\{0\} : 3m > 0\}$

$= \{m \in \mathbb{Z}-\{0\} : m > 0\} = \mathbb{Z}^+$

$[-1] = \{m \in \mathbb{Z}-\{0\} : -1Em\}$

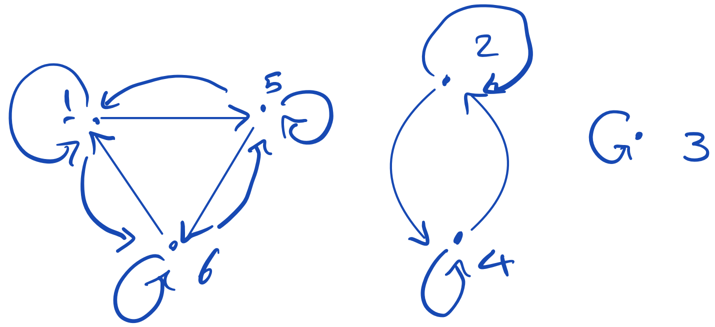
$= \{m \in \mathbb{Z}-\{0\} : -3m > 0\}$

$= \{m \in \mathbb{Z}-\{0\} : m < 0\} = \mathbb{Z}^-$

3. Let T be the equivalence relation on $A := \{1,2,3,4,5,6\}$ with equivalence classes $\{1,5,6\}$, $\{2,4\}$ and $\{3\}$.

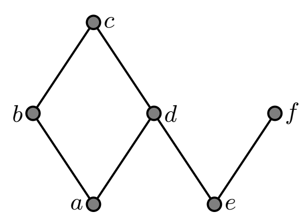
   (a) Draw the digraph of T. (1 marks)

   (b) List all ordered pairs of T. (2 marks)

a) 

b) $\{(1,1), (1,5), (1,6), (2,2), (2,4), (3,3), (4,2), (4,4), (5,1), (5,5), (5,6), (6,1), (6,5), (6,6)\}$

4. Let P be the partial ordering relation on the set $B := \{a,b,c,d,e,f\}$ represented by the following Hasse diagram.



   (a) List all ordered pairs of P. (2 marks)

   (b) Is P a total order. Justify your answer. (2 marks)

a) $\{(a,a), (a,b), (a,d), (a,c), (b,b), (b,c), (d,d), (d,c), (c,c), (e,e), (e,d), (e,c), (e,f), (f,f)\}$

b) No, a, and e are not comparable
(a,e) and (e,a) don't belong to P.

Question 3: (14 marks)

1. Consider the sets $X := \{a, b, c\}$, $Y := \{0, 1, 2\}$, and $Z := \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 0)\}$. Find the following sets.

   (a) $(X \times Y) - Z$. (1 mark)

a) $X \times Y = \{(a,0), (a,1), (a,2), (b,0), (b,1), (b,2), (c,0), (c,1), (c,2)\}$

$(X \times Y) - Z = \{(a,0), (b,0), (c,1), (c,2)\}$

   (b) $(X \cap Y) \times X$. (1 mark)

$X \cap Y = \phi$ $\quad (X \cap Y) \times X = \phi$

   (c) $\{\emptyset\} \times Z$. (1 mark)

$\{\phi\} \times Z = \{(\phi, (a,1)), (\phi, (a,2)), (\phi, (b,1)), (\phi, (b,2)), (\phi, (c,0))\}$

2. Let f be the function from $C := \{a, b, c, d\}$ to $D := \{0, 1, 2, 3, 4\}$ defined by $f(a) = 0$, $f(b) = 4$, and $f(c) = f(d) = 1$.

   (a) Find $f(\{a, b\})$ and $f(\{a, c, d\})$. (1 mark)

$f(\{a,b\}) = \{0,4\}$

$f(\{a,c,d\}) = \{0,1\}$

   (b) Find $f^{-1}(\{0, 4\})$ and $f^{-1}(\{1\})$. (1 mark)

$f^{-1}(\{0,4\}) = \{a,b\}$

$f^{-1}(\{1\}) = \{c,d\}$

   (c) Decide whether f is **one to one** or **onto**. Justify your answers. (2 marks)

Not one.to one because $c \neq d$ but $f(c) = f(d) = 1$

Not onto because $\forall x \in C \quad f(x) \neq 2$

$Rang(f) = \{0,1,4\} \neq D$.

3. Let g and h be two function from $\mathbb{R}$ to $\mathbb{R}$ defined by $g(x) = 2x - 1$ and $h(x) = 3 - 3x$.

(a) Find $g \circ h$ and $h \circ g$. (2 marks)

(b) Prove that g is a one to one correspondence function. (2 marks)

(c) Find $g^{-1}(x)$, for all $x \in \mathbb{R}$. (1 mark)

(d) Decide whether $h \circ g$ is one to one or onto. Justify your answers. (2 marks)

a) $g \circ h(x) = g(h(x)) = g(3-3x) = 2(3-3x) - 1$
$= 5 - 6x$

$h \circ g(x) = h(g(x)) = h(2x-1) = 3 - 3(2x-1)$
$= 3 - 6x + 3$
$= 6 - 6x$

b) $g: \mathbb{R} \to \mathbb{R}$ $\quad g(x) = 2x - 1$

g is one to one, because if $x_1, x_2 \in \mathbb{R}$ and $g(x_1) = g(x_2) \Rightarrow 2x_1 - 1 = 2x_2 - 1$
$\Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$

g is onto because $\forall y \in \mathbb{R}$ $\exists x = \frac{y+1}{2} \in \mathbb{R}$ such that $g(x) = 2x - 1 = 2\left(\frac{y+1}{2}\right) - 1 = y$

Hence g is one to one correspondance.

c) $g^{-1}(x) = y \Leftrightarrow x = g(y) = 2y - 1$
$\Rightarrow y = \frac{x+1}{2} \Rightarrow g^{-1}(x) = \frac{x-1}{2}$

d) $h \circ g$ is one to one correspondance

$h \circ g(x) = 6 - 6x$

$h \circ g$ is one to one:-
$\forall x_1, x_2 \in \mathbb{R}$, $h \circ g(x_1) = h \circ g(x_2) \Leftrightarrow 6 - 6x_1 = 6 - 6x_2$
$\Leftrightarrow x_1 = x_2$

$h \circ g$ is onto, $\forall y \in \mathbb{R}$ $\exists x = \frac{6-y}{6} \in \mathbb{R}$ $\ni$:
$h \circ g(x) = 6 - 6x = 6 - 6\left(\frac{6-y}{6}\right) = y$

Question 4: (3 marks)

1. Give the cardinal of each of the following sets.
   (a) $A_1 := \{k \in \mathbb{Z}; k \text{ is odd}\}$. (1 mark)
   (b) $A_2 := [0, \infty) \cap \mathbb{Q}^+$. (1 mark)
2. Show that the set $O := \{a \in \mathbb{Z}; 3|a\}$ is countable. (1 mark)

a) $A_1 \subseteq \mathbb{Z} \implies A_1$ has cardinality $\aleph_0$

b) $A_2 \subseteq \mathbb{Q}^+ \implies A_2$ is countable
$\implies |A_2| = \aleph_0$

2. $O = \{a \in \mathbb{Z} : 3|a\}$

$O = \{\ldots, -6, -3, 0, 3, 6, 9, \ldots\}$

define $f : O \to \mathbb{N}$ by

$$f(x) = \begin{cases} \frac{2x}{3} & \text{if } x > 0 \\ -\frac{2x}{3} + 1 & \text{if } x < 0 \end{cases}$$

f is 1-1 ont.

$\implies O$ is countable.

This can be proved also by $O \subseteq \mathbb{Z}$ and $\mathbb{Z}$ is countable.