Math 204
Differential Equations
Fourier Series

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Orthogonal Functions

Inner Product

Definition (Inner Product)

The inner product of two continuous functions f and g on $[a, b]$ is

$$(f, g) = \int_a^b f(x)g(x)dx$$

Orthogonal Functions

Definition (Orthogonal Product)

Two functions f and g are said to be orthogonal on $[a, b]$ is

$$(f, g) = \int_a^b f(x)g(x)dx = 0$$

Orthogonal Functions

Example

The functions $\sin x$ and $\cos x$ are orthogonal on $[-\pi, \pi]$

Orthogonal Set

Definition (Orthogonal Set)

We say that the set of functions

$$\{\phi_1(x), \phi_2(x), \dots, \}$$

is orthogonal on $[a, b]$ if

$$(\phi_m, \phi_m) = \int_a^b \phi_m(x)\phi_n(x)dx = 0, \qquad m \neq n$$

Norm

Definition (Norm)

We define the norm of the function f by

$$||f|| = \sqrt{(f,f)} = \sqrt{\int_a^b f(x)^2 \, dx}$$

Orthonormal Sets

Definition (Orthonormal Set)

If the set

$$\{\phi_1(x), \phi_2(x), \dots,\}$$

is orthogonal on $[a, b]$ and

$$||\phi_n|| = 1, \quad \text{for} \quad n = 1, 2, \dots$$

then the set is said to be orthonormal on $[a, b]$.

Example

Show that the set $\{1, \cos x, \cos 2x, ...\}$ is orthogonal on $[-\pi, \pi]$

Orthonormal Sets

Example

Find the corresponding orthonormal set from the set $\{1, \cos x, \cos 2x, ...\}$ on $[-\pi, \pi]$

Example

Find the corresponding orthonormal set from the set $\{1, \cos x, \cos 2x, \ldots\}$ on $[-\pi, \pi]$

Solution:

$$(\sin nx, \sin nx) = \int_{-\pi}^{\pi} \sin^2 nx\, dx = \tfrac{1}{2}\int_{-\pi}^{\pi} (1 - \cos 2nx)\, dx = \tfrac{1}{2}(2\pi - 0) = \pi$$

$$(\cos nx, \cos nx) = \int_{-\pi}^{\pi} \cos^2 nx\, dx = \tfrac{1}{2}\int_{-\pi}^{\pi} (1 + \cos 2nx)\, dx = \tfrac{1}{2}(2\pi + 0) = \pi$$

$$\{\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \ldots\}$$

Orthogonal Functions

Example

Show that $\{1, \sin x, \cos x, \sin 2x, \cos 2x, ...\}$ is orthogonal on $[-\pi, \pi]$

Example

Show that $\{1, \sin x, \cos x, \sin 2x, \cos 2x, ...\}$ is orthogonal on $[-\pi, \pi]$

1. Constant function with sines and cosines. For $n \geq 1$,

$$(1, \cos nx) = \int_{-\pi}^{\pi} \cos(nx)\, dx = \frac{\sin(nx)}{n}\bigg|_{-\pi}^{\pi} = \frac{\sin(n\pi) - \sin(-n\pi)}{n} = 0,$$

since $\sin(n\pi) = 0$. Similarly,

$$(1, \sin nx) = \int_{-\pi}^{\pi} \sin(nx)\, dx = -\frac{\cos(nx)}{n}\bigg|_{-\pi}^{\pi} = -\frac{\cos(n\pi) - \cos(-n\pi)}{n} = 0$$

because $\cos(n\pi) = \cos(-n\pi)$. Thus 1 is orthogonal to every $\sin nx$ and $\cos nx$ for $n \geq 1$.

Orthogonal Functions

2. Sine with sine.

$$\sin(mx)\sin(nx) = \tfrac{1}{2}\big(\cos((m-n)x) - \cos((m+n)x)\big).$$

Hence, if $m \neq n$,

$$(\sin mx, \sin nx) = \tfrac{1}{2}\int_{-\pi}^{\pi} \cos((m-n)x)\,dx - \tfrac{1}{2}\int_{-\pi}^{\pi} \cos((m+n)x)\,dx = 0$$

3. Cosine with cosine.

$$\cos(mx)\cos(nx) = \tfrac{1}{2}\big(\cos((m-n)x) + \cos((m+n)x)\big),$$

we get for $m \neq n$,

$$(\cos mx, \cos nx) = 0,$$

Even and Odd Functions

- A function f is even if $f(-x) = f(x)$
- A function f is odd if $f(-x) = -f(x)$

Example

The function $f(x) = x^2$ is even, $f(x) = x^3 + \sin x$ is odd.

- If $f(x)$ is an even continuous function on $[-L, L]$, then

$$\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx$$

- If $f(x)$ is an odd continuous function on $[-L, L]$, then

$$\int_{-L}^{L} f(x)\,dx = 0$$

Periodic Fucntion

Definition

A function f is periodic (with period L) if

$$f(x+L) = f(x), \quad L > 0$$

Fourier Series

Fourier Series

The **Fourier Series** is a way to represent a periodic function as an infinite sum of sines and cosines.
If $f(x)$ is periodic with period 2π, then it can be expressed as:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \ + b_n \sin \ nx$$

Fourier Series

If f and f' are piecewise continuous on the interval $[-\pi, pi]$, or f is define outside the interval $[-\pi, \pi]$ so that it is periodic with period 2π, then f has a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where the coefficients are given by:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx, \qquad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos{(nx)}\ dx,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin{(nx)}\ dx.$$

Definition (Fourier Series)

If f and f' are piecewise continuous on the interval $[-L, L]$, or f is define outside the interval $[-L, L]$ so that it is periodic with period $2L$, then f has a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

where the coefficients are given by:

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\,dx, \qquad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right)\,dx,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right)\,dx.$$

Convergence of Fourier Series

The Fourier series converges to $f(x)$ if f is continuous at x, and to

$$\frac{f(x^+) + f(x^-)}{2}$$

if f is discontinuous at x.
For $x = L$ and $x = -L$, the series converges to

$$\frac{f(-L^+) + f(L^-)}{2}$$

Symmetric Functions

- If $f(x)$ is even, then Fourier coefficients

$$a_0 = \frac{2}{L} \int_0^L f(x)\, dx,$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2, ...$$

$$b_n = 0, \quad n = 1, 2, ...$$

- If $f(x)$ is odd, then Fourier coefficients

$$a_0 = 0, \quad n = 0, 1, 2, ...$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2, ...$$

Example

Find the Fourier series to the function

$$f(x) = |x|, \quad |x| \leq L$$

$$f(x + 2L) = f(x)$$

Then find the value of

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

Fourier Series

Example

Find the Fourier series to the function

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$$f(x + 2L) = f(x)$$

Then find the value of

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

Solution:

$$f(x) = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{L}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

Example

Find the Fourier series to the function

$$f(x) = x - x^2, \quad x \in [-\pi, \pi]$$

Then deduce that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

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Find the Fourier series to the function

$$f(x) = x - x^2, \quad x \in [-\pi, \pi]$$

Then deduce that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

Solution:

$$f(x) = \frac{-1}{3}\pi^2 - 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx - 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx$$

Example

Find the Fourier series to the function

$$f(x) = \frac{1}{4}(\pi - x)^2, \quad x \in (0, 2\pi)$$

Then deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Fourier Series

Example

Find the Fourier series to the function

$$f(x) = \frac{1}{4}(\pi - x)^2, \quad x \in (0, 2\pi)$$

Then deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Solution:

$$f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$$

Fourier Series

Example

Find the Fourier series to the function

$$f(x) = \begin{cases} 0, & -3 < x < 0, \\ 3, & 0 < x < 3 \end{cases}$$

where $f(x+6) = f(x)$

Example

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$$f(x) = \begin{cases} 0, & -3 < x < 0, \\ 3, & 0 < x < 3 \end{cases}$$

where $f(x+6) = f(x)$

Solution:

$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin \frac{(2n-1)\pi x}{3}$$

The series converges to $\frac{3}{2}$ for $x = -3, 0, 3$

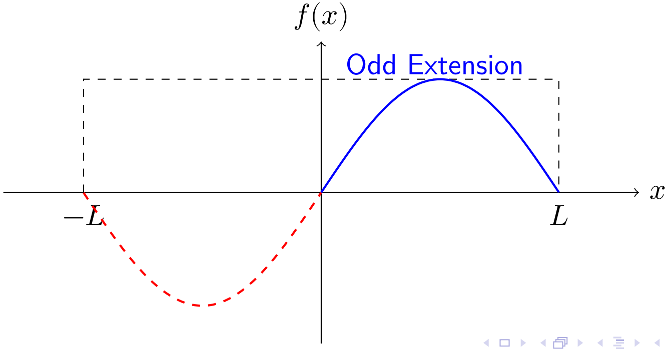
Fourier Cosine and Sine Series

Fourier Cosine and Sine Series

If a function f is defined on $(0, L)$, and we want to represent this function by a trigonometric series, then we can extend $f(x)$ to an odd function f_o

$$f_o(x) = \begin{cases} f(x), & 0 < x < L, \\ -f(-x), & -L < x < 0. \end{cases}$$

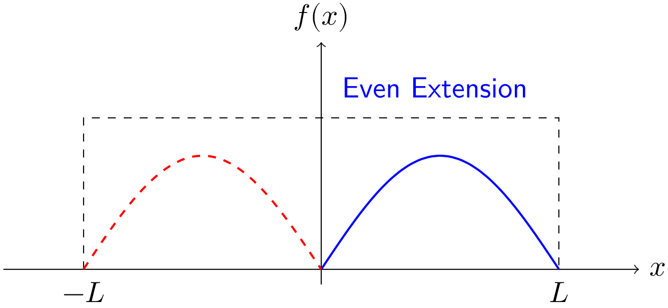
with $f_o(x + 2L) = f_o(x)$ which can be approximated by $\sin$ series



we can extend $f(x)$ to an even function f_e

$$f_e(x) = \begin{cases} f(x), & 0 < x < L, \\ f(-x), & -L < x < 0. \end{cases}$$

with $f_e(x+2L) = f_e(x)$ which can be approximated by $\cos$ series



Fourier Cosine Series

Definition

Let $f(x)$ be piecewise continuous function on the interval $[0, L]$. The Fourier cosine series of f on $[0, L]$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L},$$

where

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx, \qquad n = 0, 1, 2, \ldots$$

Fourier Sine Series

Definition

Let $f(x)$ be piecewise continuous function on the interval $[0, L]$. The Fourier sine series of f on $[0, L]$ is

$$\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L},$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx, \qquad n = 1, 2, \ldots$$

Example

Compute the Fourier sine series for

$$f(x) = \cos\frac{\pi x}{3}, \qquad 0 < x < 3$$

Example

Compute the Fourier cosine series for

$$f(x) = \sin x, \qquad 0 < x < \pi/2$$

and deduce that

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{2}$$

Example

Compute the Fourier cosine series for

$$f(x) = \sin x, \qquad 0 < x < \pi/2$$

and deduce that

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{2}$$

Solution

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \cos 2nx$$

Example

* Find the Fourier series for

$$f(x) = \begin{cases} -\sin x, & -\pi/2 < x < 0, \\ \sin x, & 0 < x < \pi/2 \end{cases}$$

and deduce that

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{2}$$

Fourier Cosine and Sine Series

Example

Find the Fourier sine series for

$$f(x) = \begin{cases} x, & 0 \le x \le \pi/2, \\ \pi - x, & \pi/2 < x \le \pi \end{cases}$$

Example

Find the Fourier sine series for

$$f(x) = \begin{cases} x, & 0 \leq x \leq \pi/2, \\ \pi - x, & \pi/2 < x \leq \pi \end{cases}$$

Solution

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin(2n-1)x$$

At $x = \pi$ and $x = -\pi$ the series converges to 0

Fourier Integral

Fourier Integral

Definition

The Fourier Integral of a function f defined on $(-\infty, \infty)$ is given by

$$f(x) = \frac{1}{\pi} \int_0^{\infty} [A(\lambda)\cos(\lambda x) + B(\lambda)\sin(\lambda x)]\, d\lambda$$

$$A(\lambda) = \int_{-\infty}^{\infty} f(t)\cos(\lambda t)\, dt$$

$$B(\lambda) = \int_{-\infty}^{\infty} f(t)\sin(\lambda t)\, dt$$

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t)\cos(\lambda t - \lambda x)\, dt\, d\lambda$$

Fourier Integral

Theorem

If f is absolutely integrable

$$\int_{-\infty}^{\infty} |f(x)|dx < \infty$$

and f, f' are piecewise continuous on every finite interval, then the Fourier integral of f converges to $f(x)$ at a point of continuity and converges to

$$\frac{f(x^{+}) + f(x^{-})}{2}$$

at a point of discontinuity.

Fourier Integral

Example

Compute the Fourier integral of the function

$$f(x) = \begin{cases} 1, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$

and evaluate $\displaystyle\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda$ and $\displaystyle\int_0^\infty \frac{\sin\lambda}{\lambda}\,d\lambda$

Fourier Integral

Example

Compute the Fourier integral of the function

$$f(x) = \begin{cases} 1, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$

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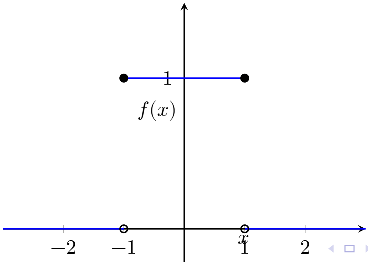

Example

Compute the Fourier integral of the function

$$f(x) = \begin{cases} 0, & x < -\pi \\ -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \\ 0, & \pi < x \end{cases}$$

Fourier Sine and Cosine Integrals

Fourier Sine and Cosine Integrals

If f is an odd function on $(-\infty, \infty)$, then

$$A(\lambda) = \int_{-\infty}^{\infty} f(t)\cos(\lambda t)\,dt = 0$$

$$B(\lambda) = \int_{-\infty}^{\infty} f(t)\sin(\lambda t)\,dt = 2\int_{0}^{\infty} f(t)\sin(\lambda t)\,dt$$

If f is an even function on $(-\infty, \infty)$, then

$$A(\lambda) = \int_{-\infty}^{\infty} f(t) \cos(\lambda t)\, dt = 2 \int_{0}^{\infty} f(t) \cos(\lambda t)\, dt$$

$$B(\lambda) = \int_{-\infty}^{\infty} f(t) \sin(\lambda t)\, dt = 0$$

Example

Compute the Fourier integral of the function

$$f(x) = \begin{cases} |\sin x|, & |x| \leq \pi \\ 0, & |x| > \pi \end{cases}$$

and deduce that

$$\int_0^\infty \frac{\cos \lambda\pi + 1}{1 - \lambda^2} \cos(\frac{\pi\lambda}{2})\, d\lambda = \frac{\pi}{2}$$