Finite Abelian Groups - Tutorial

Exercise 1: What is the smallest positive integer n such that there are two non-isomorphic groups of order n? Name the two groups.

Solution: There is only one group upto isomorphism of $n = 1, 2$, or 3**.** For**** $n = 4$. The two groups are $\mathbb{Z}_4$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Exercise 3: What is the smallest positive integer n such that there are exactly four non-isomorphic Abelian groups of order n? Name the four groups.

Solution: n would have to be of the form $p^2 q^2$. The smallest would be $n = 2^{23^2} = 36$. The four groups are:

- $\mathbb{Z}_9 \oplus \mathbb{Z}_4$
- $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4$
- $\mathbb{Z}_9 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$
- $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

Exercise 5: Prove that any Abelian group of order 45 has an element of order 15. Does every Abelian group of order 45 have an element of order 9?

Solution: The only Abelian groups of order 45 are $\mathbb{Z}_{45}$ and $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$.

- In $\mathbb{Z}_{45}$, the element 3 has order $|3| = 15$
- In $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$, the element $(1, 1, 1)$ has order $|(1, 1, 1)| = \text{lcm}(3, 3, 5) = 15$

Therefore every Abelian group of order 45 has an element of order 15.

However, $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$ does not have an element of order 9.

Exercise 6: Show that there are two Abelian groups of order 108 that have exactly one subgroup of order 3.

Solution: Since $108 = 27 \times 4 = 3^3 \times 2^2$, we examine the order-27 component. By Theorem $\mathbb{Z}_{27}$ one subgroup order 3. The two Abelian groups of order 108 with exactly one subgroup of order 3 are:

- $\mathbb{Z}_{27} \oplus \mathbb{Z}_4$
- $\mathbb{Z}_{27} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

Exercise 9: Suppose that G is an Abelian group of order 120 and that G has exactly three elements of order 2. Determine the isomorphism class of G.

Solution: Since $120 = 8 \times 3 \times 5 = 2^3 \times 3 \times 5$, elements of order 2 are determined by the 2-power factors in the direct product. By Theorem:

- $\mathbb{Z}_8$ has exactly 1 element of order 2
- $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ has exactly 3 elements of order 2
- $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ has exactly 7 elements of order 2

Since G has exactly 3 elements of order 2, the 2-power component must be $\mathbb{Z}_4 \oplus \mathbb{Z}_2$.

Therefore, $G \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$.

Exercise 12: Suppose that the order of some finite Abelian group is divisible by 10. Prove that the group has a cyclic subgroup of order 10.

Solution: Since $10 = 2 \times 5$ is square-free (each prime factor occurs to the first power only), by the Fundamental Theorem of Finite Abelian Groups, the group can be written as a direct product of cyclic groups of prime-power order.

Since the order is divisible by 10, the group contains factors with orders divisible by 2 and by 5. Let a be an element in a factor whose order is divisible by 2, and b be an element in a factor whose order is divisible by 5, chosen so that $|a| = 2$ and $|b| = 5$.

Since $\gcd(2, 5) = 1$, the element ab (or (a, b) in external direct product notation) has order $\mathrm{lcm}(2, 5) = 10$.

Thus the group has a cyclic subgroup $\langle ab \rangle$ of order 10.

Exercise 15(a): How many Abelian groups (up to isomorphism) are there of order 6?

Solution: Since $6 = 2 \times 3$ and both prime factors occur to the first power, 6 is not divisible by the square of any prime. By the Fundamental Theorem, there is exactly **1** Abelian group of order 6 up to isomorphism, namely $\mathbb{Z}_6$.

Exercise 15(b): How many Abelian groups (up to isomorphism) are there of order 15?

Solution: Since $15 = 3 \times 5$ and both prime factors occur to the first power, 15 is not divisible by the square of any prime. By the Fundamental Theorem, there is exactly **1** Abelian group of order 15 up to isomorphism, namely $\mathbb{Z}_{15}$.

Exercise 15(d): How many Abelian groups (up to isomorphism) are there of order pq, where p and q are distinct primes?

Solution: Since pq is the product of two distinct primes, neither prime occurs to a power greater than 1. Therefore pq is not divisible by the square of any prime. By the Fundamental Theorem, there is exactly **1** Abelian group of order pq up to isomorphism, namely $\mathbb{Z}_{pq}$.

Exercise 21: The set $\{1, 9, 16, 22, 29, 53, 74, 79, 81\}$ is a group under multiplication modulo 91. Determine the isomorphism class of this group.

Solution: The group is Abelian and has order 9. The only possibilities are $\mathbb{Z}_9$ and $\mathbb{Z}_3 \oplus \mathbb{Z}_3$.

$\mathbb{Z}_9$ has exactly 2 elements of order 3, while $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ has 8 elements of order 3. Computing orders: $|9| = |16| = |22| = 3$. Since there are at least three elements of order 3, the group cannot be $\mathbb{Z}_9$. Therefore, the group is isomorphic to $\mathbb{Z}_3 \oplus \mathbb{Z}_3$.

Exercise 28: The set $G = \{1, 4, 11, 14, 16, 19, 26, 29, 31, 34, 41, 44\}$ is a group under multiplication modulo 45. Write G as an external and an internal direct product of cyclic groups of prime-power order.

Solution: Since $|G| = 12 = 4 \times 3 = 2^2 \times 3$, so only two possibilities $\mathbb{Z}_4 \oplus \mathbb{Z}_3$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$ as an external direct product. since 19 and 44 both have order 2, so G cannot be cyclic. Hence G must be isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$.

One internal direct product representation is: $G = \langle 19 \rangle \times \langle 44 \rangle \times \langle 31 \rangle$

where $|19| = 2$, $|44| = 2$, and $|31| = 3$.