M - 107

SECOND MID-TERM EXAMSEMESTER I, (1437 -1438)

Question: 1. (a) Given vectors

$u = <3,-2,2>, v = <4,5,7>$ and $w = <0,1,3>$. Find

(i) $u.(v+w)$ (ii) $Comp_w v$ (iii) $Proj_v u$ [9]

(2) (i) $v + w = <4, 6, 10>$

$u \cdot (v+w) = 12 - 12 + 26 = 26$

(3) (ii) $Comp_w v = \frac{v \cdot w}{\|w\|}$ $v \cdot w = 0 + 5 + 21 = 26$

$\|w\| = \sqrt{0+1+9} = \sqrt{10}$

$= \frac{26}{\sqrt{10}}$

(4) (iii) $Proj_v u = (Comp_v u)\frac{v}{\|v\|}$ $u.v = 16$

$\|v\| = \sqrt{90}$

$= \frac{16}{\sqrt{90}} \cdot \frac{1}{\sqrt{90}} <4,5,7> = \frac{16}{90} <4,5,7>$

$= \frac{8}{45} <4,5,7>$

(b) Determine whether the lines

$l_1: x = 1+2t, y = 1-4t, z = 5-t$

$l_2: x = 4-v, y = -1+6v, z = 4+v$ [9]

are intersecting or parallel. If intersecting, find the point of intersection, also find the angle between the lines.

(2) i. vector parallel to line l_1 $a = <2,-4,-1>$

Vector parallel to line l_2 $b = <-1, 6, 1>$

$a \not\parallel b \Rightarrow l_1 \not\parallel l_2$

(5) ii. $1+2t_0 = 4 - v_0$, $1 - 4t_0 = -1 + 6v_0$ $\Rightarrow$ $t_0 = 2$, $v_0 = -1$

$5 - t_0 = 4 + v_0$ satisfy third Equation $\Rightarrow$ lines intersects

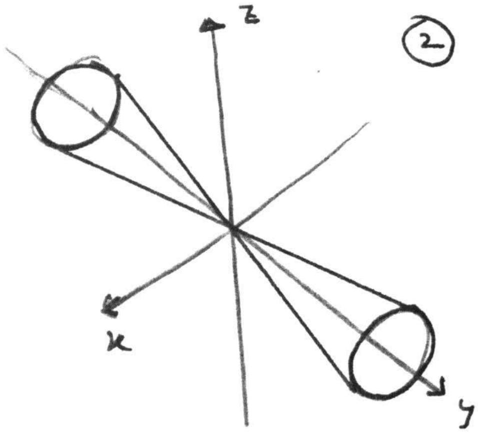
Point of intersection $(x,y,z) = (5,-7,3)$

(2) (iii) $\cos\theta = \frac{a \cdot b}{\|a\|\|b\|} = \frac{-27}{\sqrt{21}\sqrt{38}}$

$\theta = \cos^{-1}\left(\frac{-27}{\sqrt{21}\sqrt{38}}\right)$

Question: 2 . (a) Identify the surface $36x^2 - 16y^2 + 9z^2 = 0$. **Find its traces on the coordinate planes and then sketch the surface.** **[6]**

(1) - Surface is cone



- Trace   Eqn.   discr.

(3)

Trace	Eqn.	discr.
yz	$y = \pm\frac{3}{4}z$	intersecting lines
xy	$y = \pm\frac{6}{4}x$	intersecting lines
xz	$36x^2 + 9z^2 = 0$ (0, 0, 0)	Point.
$y \neq 0$		Ellipse.

(b) Let the curve be determined by the function
$r(t) = \langle 2\cos t, 2\sin t, 3 \rangle, \quad t \in R$. **[8]**

Find Unit tangent vector T(t),
Principal normal vector N(t) and
equation of tangent vector at the point P on the curve C where $t = \frac{\pi}{2}$.

$r' = \langle -2\sin t, 2\cos t, 0 \rangle \qquad \|r'(t)\| = 2$

(2) 1. Unit tangent vector $T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{1}{2}\langle -2\sin t, 2\cos t, 0 \rangle$

(2) 2. Principal Normal vector $N(t) = \frac{T'(t)}{\|T'(t)\|}$ $\qquad = \langle -\sin t, \cos t, 0 \rangle$

$T'(t) = \langle -\cos t, -\sin t, 0 \rangle \qquad , \|T'(t)\| = 1$

$N(t) = \langle -\cos t, -\sin t, 0 \rangle$

(2) 3. Equation of Tangent line. $\quad T(\frac{\pi}{2}) = \langle -1, 0, 0 \rangle$

Point $P|_{t=\frac{\pi}{2}} = (0, 2, 3)$

$x = 0 - 2t, \quad y = 2, \quad z = 3, \quad t \in R$

(2) 4. Equation of normal line $\quad N(\pi/2) = \langle 0, -1, 0 \rangle, \quad P|_{t=\frac{\pi}{2}} = (0, 2, 3)$

$x = 0, \quad y = 2 - s, \quad z = 3, \quad s \in R$

Question: 3. (a) Suppose acceleration of a point moving along the curve at time t is given by $a(t) = i + 2tj + 3t^2k$ and its velocity at t = 0 is $v(0) = <0, 1, -1>$.
Find the tangential and normal components of acceleration and the curvature of the curve C at t = 1. **[10]**

$r''(t) = a(t) = i + 2tj + 3t^2k$

$v(t) = r'(t) = \int (i + 2tj + 3t^2k)\,dt = ti + t^2j + t^3k + c$

$v(0) = <0, 1, -1> = 0 + c \Rightarrow c = j - k$

(3) $r'(t) = ti + (t^2 + 1)j + (t^3 - 1)k$.

$r'(1) = <1, 2, 0>$, $r''(1) = <1, 2, 3>$. $\|r'(1)\| = \sqrt{1+4} = \sqrt{5}$

$r'(1) \times r''(1) = <6, -3, 0>$, $\|r'(1) \times r''(1)\| = 3\sqrt{5}$

$r'(1) \cdot r''(1) = 5$

(3) $a_T = \dfrac{r'(1) \cdot r''(1)}{\|r'(1)\|} = \dfrac{5}{\sqrt{5}} = \sqrt{5}$

(3) $a_N = \dfrac{\|r'(1) \times r''(1)\|}{\|r'(1)\|} = \dfrac{3\sqrt{5}}{\sqrt{5}} = 3$

(1) $\kappa = \dfrac{a_N}{\|r'(1)\|^2} = \dfrac{3}{(\sqrt{5})^2} = \dfrac{3}{5}$

(b) Find radius and center of curvature to the curve $r(t) = t\ i + t^2\ j$ at the point P(-1,1). **[8]**

(i) $\kappa\Big|_{t=-1} = \dfrac{|f'g'' - g'f''|}{[f'^2 + g'^2]^{3/2}} = \dfrac{2}{5^{3/2}}$

(4) Radius of curvature $\rho = \dfrac{1}{\kappa} = \dfrac{5^{3/2}}{2}$

(ii) center of curvature (h, k)

$x = t,\ y = t^2 \Rightarrow y = x^2,\ y' = 2x,\ y'' = 2$

$y\big|_p = 1,\ y'\big|_p = -2,\ y''\big|_p = 2$

(4) $h = x_p - \dfrac{y'(1+y'^2)}{y''} = -1 + \dfrac{10}{2} = \dfrac{8}{2} = 4$

$k = y_p + \dfrac{1+y'^2}{y''} = 1 + \dfrac{5}{2} = \dfrac{7}{2}$

center of curvature $(h, k) = (4, 7/2)$