Lecture 3.1: Algebra of matrix

Basic definitions of matrices are given in Lecture 1.

3.1.1 Properties of a matrix

1. **Transpose of a Matrix:** A transpose of a matrix is obtained by interchanging rows and corresponding columns of the given matrix. The transpose of the matrix A is denoted A^t .

$$A=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}, \quad A^t=\begin{bmatrix}1 & 4\\ 2 & 5\\ 3 & 6\end{bmatrix}$$

Properties of the Transpose of a matrix

1. $(A^t)^t = A$
2. $(AB)^t = B^tA^t$
3. $(kA)^t = kA^t$, where k is a scalar.
4. $(A+B)^t = A^t + B^t$

2. Symmetric Matrix:
A square matrix is symmetric if $A^t = A$.

$$A=\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 5\\ 3 & 5 & 6\end{bmatrix}, \quad A^t=\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 5\\ 3 & 5 & 6\end{bmatrix}, \quad A^t = A$$

3. Skew – symmetric Matrix :
A square matrix is skew symmetric if $A^t = - A$.

$$A=\begin{bmatrix}0 & -2 & -3\\ 2 & 0 & -4\\ 3 & 4 & 0\end{bmatrix}, \quad A^t=\begin{bmatrix}0 & 2 & 3\\ -2 & 0 & 4\\ -3 & -4 & 0\end{bmatrix}, \quad A^t=-A.$$

4. Equality of matrix:
Two matrices are equal, if these of same size and corresponding entries are equal.

$$A=\begin{bmatrix}2 & 1\\ 3 & 4\end{bmatrix}, \quad B=\begin{bmatrix}2 & 1\\ 3 & 4\end{bmatrix}$$

A and B are equal matrices when these of the same size and corresponding entries are equal.

Example:1. Write down the system of equation, if matrices A and B are equal

$$A = \begin{bmatrix} x-2 & y-3 \\ x+y & z+3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3+z \\ z & y \end{bmatrix}$$

Solution: A and B are of the same size, hence

$$A = B \Rightarrow$$

$$\begin{aligned} x-2 &= 1 \\ y-3 &= 3+z \\ x+y &= z \\ z+3 &= y \end{aligned}$$

System of equations are

$$\begin{aligned} x \qquad &= 3 \\ y - z &= 6 \\ x + y - z &= 0 \\ -y + z &= -3 \end{aligned}$$

3.1.2 Addition of matrices:

Matrices of the equal size can be added entry wise.

Example:2. Add the following matrices:

$$\begin{bmatrix} 1 & 0 & 2 \\ 3 & 5 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 2 & 8 \\ 2 & 4 & 1 \end{bmatrix}$$

Solution. We need to add the pairs of entries, and then simplify for the final answer:

$$\begin{bmatrix} 1 & 0 & 2 \\ 3 & 5 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 2 & 8 \\ 2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1+4 & 0+2 & 2+8 \\ 3+2 & 5+4 & 4+1 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 10 \\ 5 & 9 & 5 \end{bmatrix}$$

So the answer is:

$$\begin{bmatrix} 5 & 2 & 10 \\ 5 & 9 & 5 \end{bmatrix}$$

Example:3. Find the value of x and y in the following matrix equation

$$\begin{bmatrix} 5 & x \\ 3y & 2 \end{bmatrix} + \begin{bmatrix} -3 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 5 & 7 \end{bmatrix}$$

Solution. Using concept of addition of matrices, we simplify left hand side

$$\begin{bmatrix} 5-3 & x+2 \\ 3y-1 & 2+5 \end{bmatrix} = \begin{bmatrix} 2 & x+2 \\ 3y-1 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 5 & 7 \end{bmatrix}$$

Two matrices are equal when their correspoding entries are equal

x + 2 = 4

2y – 1 = 5

Solving these equations

x = 4 – 2 =2

3y = 5+1

3y = 6,   y = 2

Solution of matrix equation is x = 2, y = 2.

3.1.3 Scalar Multiplication:

If a matrix is multiplied by a scalar α, then each entry is multiplied by scalar α.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}, \quad 2A = 2\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}, \quad 2A = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 2 & 0 \\ 2 & 2 & 4 \end{bmatrix}$$

$$3A = \begin{bmatrix} 3 & 6 & 9 \\ 6 & 3 & 0 \\ 3 & 3 & 6 \end{bmatrix}$$

3.1.4 Matrix Multiplication:

The product of two matrices A and B is possible if the number of columns of A is equal to number of rows in B, the method is being explained by following example:

$$A=\begin{bmatrix}1 & 2 & 4\\ 2 & 6 & 0\end{bmatrix}_{2x3}, \quad B=\begin{bmatrix}4 & 1 & 4 & 3\\ 0 & -1 & 3 & 1\\ 2 & 7 & 5 & 2\end{bmatrix}_{3X4}$$

A x B = C

2x3 3x4 2x4

$$AB=\begin{bmatrix}c_{11} & c_{12} & c_{13} & c_{14}\\ c_{21} & c_{22} & c_{23} & c_{24}\end{bmatrix}$$

c_{11} = 1x4 + 2x0 + 4x2 = 4 + 0 + 8 = 12
c_{12} = 1x1 + 2x(-1) + 4x7 = 1 - 2 + 28 = 27
c_{13} = 1x4 + 2x3 + 4x5 = 4 + 6 + 20 = 30
c_{14} = 1x3 + 2x1 + 4x2 = 3 + 2 + 8 = 13
c_{21} = 2x4 + 6x0 + 0x2 = 8 + 0 + 0 = 8
c_{22} = 2x1 + 6x(-1) + 0x7 = 2 -6 + 0 = -4
c_{23} = 2x4 + 6x3 + 0x5 = 8 + 18 + 0 = 26
c_{24} = 2x3 + 6x1 + 0x2 = 6 + 6 + 0 = 12

$$AB=\begin{bmatrix}12 & 27 & 30 & 13\\ 8 & -4 & 26 & 12\end{bmatrix}$$

NOTE: AB $\neq$ BA

Lecture 3.2 : Inverse of matrix and power of matrix

3.2.1 Inverse of a 2x2 matrix

Consider a 2x2 matrix A = $\begin{bmatrix}a & b\\ c & d\end{bmatrix}$

If ad – bc $\neq$ 0, then $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$

Note: Multiple $(ad-bc)$ is called the **determinant** of matrix A = $\begin{bmatrix}a & b\\ c & d\end{bmatrix}$

Example: Find inverse of matrix A = $\begin{bmatrix}3 & 2\\ 4 & 5\end{bmatrix}$

ad - bc = 3x5 –2x4 =15 – 8 = 7

$$A^{-1}=\tfrac{1}{7}\begin{bmatrix}5 & -2\\ -4 & 3\end{bmatrix}.$$

Properties of Inverse

1. $A^{-1}A = A\,A^{-1} = I$
2. If A and B are invertible matrices of the same size , then AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$

3.2.2 Power of a matrix

1. $A^0 = I$
2. $A^n = A.A.A\ldots A$ (n-factors), where n>0.
3. $A^{-n} = \left(A^{-1}\right)^n = A^{-1}.A^{-1}.A^{-1}....A^{-1}$ (n- factors) , where n>0.
4. $A^rA^s = A^{r+s}$
5. $(A^r)^s = A^{rs}$
6. $(A^{-1})^{-1} = A$
7. $(A^n)^{-1} = (A^{-1})^n, \quad n = 0,1,2,...$
8. $(kA)^{-1} = \dfrac{1}{k}A^{-1}$, where k is a scalar.

Example:4. Let A be an invertible matrix and suppose that inverse of 7A is $\begin{bmatrix} -2 & 7 \\ 1 & -3 \end{bmatrix}$, find matrix A

Solution: $(7A)^{-1} = \dfrac{1}{7}A^{-1} = \begin{bmatrix} -2 & 7 \\ 1 & -3 \end{bmatrix}$

$$A^{-1} = 7\begin{bmatrix} -2 & 7 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} -14 & 49 \\ 7 & -21 \end{bmatrix}$$

$$A = (A^{-1})^{-1} = -\frac{1}{49}\begin{bmatrix} -21 & -49 \\ -7 & -14 \end{bmatrix} = \frac{7}{49}\begin{bmatrix} 3 & 7 \\ 1 & 2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 3 & 7 \\ 1 & 2 \end{bmatrix}$$

.

Example:5. Let A be a matrix $\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$ compute A^3, A^{-3}, $A^2 - 2A + I$.

Solution:

$$A^2 = AA = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 12 & 1 \end{bmatrix}$$

$$A^3 = A^2A = \begin{bmatrix} 4 & 0 \\ 12 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 28 & 1 \end{bmatrix}$$

$$A^{-3} = (A^3)^{-1} = \frac{1}{8}\begin{bmatrix} 1 & 0 \\ -28 & 8 \end{bmatrix}$$

$$A^2 - 2A + I = \begin{bmatrix} 4 & 0 \\ 12 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 8 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 0 \end{bmatrix}$$

Example:6. Find inverse of the matrix

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Solution:

ad – bc = $\cos^2\theta + \sin^2\theta = 1$,

$$A^{-1} = \frac{1}{1}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Lecture 3.3 Inverse by Elementary Matrix

3.3.1 Elementary Matrix

An nxn matrix is called *elementary matrix*, if it can be obtained from nxn identity matrix by performing a single elementary row operation.

Examples: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a 3x3 identity matrix.

Elementary matrices E_1,E_2 and E_3 can be obtained by single row operation.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \text{-3R}_3$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -2 \end{bmatrix} \text{-2R}_3\text{+R}_2$$

$$E_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

NOTE:

When a matrix A is multiplied from the left by an elementary matrices E, the effect is same as to perform an elementary row operation on A.

Example: 1.

Let A be a 3x4 matrix, $A=\begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}$ and

E be 3x3 elementary matrix obtained by row operation $3R_1+R_3$ from an Identity matrix

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

$$EA=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 9 \end{bmatrix}, 3R_1+R_3 .$$

3.3.2 Method for finding Inverse of a matrix

To find the inverse of an invertible matrix, we must find a sequence of elementary row operations that reduces A to the identity and then perform this same sequence of operations on I_n to obtain A^{-1}.

$$\left[A \mid I\right] \text{ to } \left[I \mid A^{-1}\right]$$

Example:2. Find inverse of a matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}$ by using Elementary matrix method.

Solution:

$$\left[A|I\right] = \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right]$$

$$\approx \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right] \quad -2R_1 + R_2$$

$$\approx \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right] \quad -R_2$$

$$\approx \left[\begin{array}{cc|cc} 1 & 0 & -7 & 4 \\ 0 & 1 & 2 & -1 \end{array}\right] \quad -4R_2 + R_1$$

$$= \left[I|A^{-1}\right]$$

$$A^{-1} = \begin{bmatrix} -7 & 4 \\ 2 & -1 \end{bmatrix}$$

Example:3. Use Elementary matrix method to find inverses of

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix} \qquad \text{if A is invertible.}$$

Solution:

$$[A|I] = \left[\begin{array}{ccc|ccc} 3 & 4 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right]$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 3 & 4 & -1 & 1 & 0 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right] R_1 \leftrightarrow R_2$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 4 & -10 & 1 & -3 & 0 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array}\right] -3R_1 + R_2, -2R_1 + R_3$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 4 & -10 & 1 & -3 & 0 \\ 0 & 1 & 0 & -1 & -2 & 1 \end{array}\right] -R_2 + R_3$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & \frac{1}{2} & \frac{-7}{10} & \frac{-2}{5} \end{array}\right] \quad R_2 \leftrightarrow R_3, \frac{(-4R_3 + R_2)}{10}$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{2} & \frac{-11}{10} & \frac{-6}{5} \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & \frac{-1}{2} & \frac{7}{10} & \frac{2}{5} \end{array}\right] \quad -3R_3 + R_1, -R_3$$

$$\approx [I|A^{-1}]$$

$$A^{-1} = \begin{bmatrix} \frac{3}{2} & \frac{-11}{10} & \frac{-6}{5} \\ -1 & 1 & 1 \\ \frac{-1}{2} & \frac{7}{10} & \frac{2}{5} \end{bmatrix}.$$

Lecture 4.1: Solving Linear system by Inverse Matrix

Let a given linear system of equations is

$$AX = B$$

Find A^{-1}

Multiply with A^{-1} from left

$$A^{-1}AX = A^{-1}B$$

$$I\,X = A^{-1}B$$

$$X = A^{-1}B \quad \text{is a solution.}$$

Note: To find A^{-1} we use *Elementary Matrix method*.

Example1.

Write the system of equations in a matrix form, find A^{-1}, use A^{-1} to solve the system

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 4 \\ 2x_1 + 2x_2 + x_3 &= -1 \\ 2x_1 + 3x_2 + x_3 &= 3 \end{aligned}$$

Solution: 1. Matrix Form is:

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} \quad \text{is in form of } AX = B$$

2. Find A^{-1} by using Elementary Matrix method

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\approx\left[\begin{array}{ccc|ccc}1 & 3 & 1 & 1 & 0 & 0\\ 2 & 2 & 1 & 0 & 1 & 0\\ 2 & 3 & 1 & 0 & 0 & 1\end{array}\right]$$

$$\approx\left[\begin{array}{ccc|ccc}1 & 3 & 1 & 1 & 0 & 0\\ 0 & -4 & -1 & -2 & 1 & 0\\ 0 & -3 & -1 & -2 & 0 & 1\end{array}\right] \quad \mathrm{R}_2 - 2R_1, R_3 - 2R_1$$

$$\approx\left[\begin{array}{ccc|ccc}1 & 3 & 1 & 1 & 0 & 0\\ 0 & -4 & -1 & -2 & 1 & 0\\ 0 & 0 & -1 & -2 & -3 & 4\end{array}\right] \quad 4R_3 - 3R_2$$

$$\approx\left[\begin{array}{ccc|ccc}1 & 3 & 0 & -1 & -3 & 4\\ 0 & -4 & 0 & 0 & 4 & -4\\ 0 & 0 & -1 & -2 & -3 & 4\end{array}\right] \quad R_1 + R_2, R_2 - R_3$$

$$\approx\left[\begin{array}{ccc|ccc}1 & 3 & 0 & -1 & -3 & 4\\ 0 & 1 & 0 & 0 & -1 & 1\\ 0 & 0 & -1 & -2 & -3 & 4\end{array}\right] \quad -\tfrac{1}{4}R_2, -R_3$$

$$\approx\left[\begin{array}{ccc|ccc}1 & 3 & 0 & -1 & -3 & 4\\ 0 & 1 & 0 & 0 & -1 & 1\\ 0 & 0 & -1 & -2 & -3 & 4\end{array}\right] \quad -\tfrac{1}{4}R_2, -R_3$$

$$\approx\left[\begin{array}{ccc|ccc}1 & 0 & 0 & -1 & 0 & 1\\ 0 & 1 & 0 & 0 & -1 & 1\\ 0 & 0 & -1 & -2 & -3 & 4\end{array}\right] \quad -3R_2 + R_3$$

$$\equiv \left[\mathrm{I}\,\middle|\,\mathrm{A}^{-1}\right]$$

$$\mathrm{A}^{-1} = \begin{bmatrix}-1 & 0 & 1\\ 0 & -1 & 1\\ 2 & 3 & -4\end{bmatrix}$$

$$\mathrm{X} = \mathrm{A}^{-1}\mathrm{B} = \begin{bmatrix}-1 & 0 & 1\\ 0 & -1 & 1\\ 2 & 3 & -4\end{bmatrix}\begin{bmatrix}4\\ -1\\ 3\end{bmatrix} = \begin{bmatrix}-1\\ 4\\ -7\end{bmatrix}$$

Solution set is $x_1 = -1, x_2 = 4, x_3 = -7.$

Lecture 4.2 Determinant

4.1 Determinant of a matrix

The determinant is a useful value that can be computed from the elements of a square matrix. The **determinant** of a matrix A is denoted det(A), det A, or | A|.

4.2 Evaluation of determinant of Matrix

1.The determinant of a (1×1) matrix $A = [\ a\ ]$ is just $\det A = a$.

2. The determinant of 2x2 matrix is defined as

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$|A| = \det A = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - cb$$

Example:1. Find determinant of matrix

$$A = \begin{bmatrix} 4 & 5 \\ 3 & 6 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 4 & 5 \\ 3 & 6 \end{bmatrix}$$

$$\begin{aligned} \det A &= 4x6 - 3x5 \\ &= 24 - 15 \\ &= 9 \end{aligned}$$

4.3 The determinant of 3x3 matrix is defined as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det A = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example:2. Find determinant of matrix

$$A = \begin{bmatrix} 2 & 4 & 5 \\ 3 & 6 & 8 \\ 4 & 5 & 9 \end{bmatrix}$$

Solution:

Expanding along the top row and noting alternating signs $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

$$\det A = +2x\begin{vmatrix} 6 & 8 \\ 5 & 9 \end{vmatrix} - 4x\begin{vmatrix} 3 & 8 \\ 4 & 9 \end{vmatrix} + 5x\begin{vmatrix} 3 & 6 \\ 4 & 5 \end{vmatrix}$$

$$= 2x(54-40) - 4x(27-32) + 5x(15-24)$$

$$= 2x(14) - 4x(-5) + 5x(-9)$$

$$= 28 + 20 - 45 = 48 - 45 = 3$$

Note: we can write determinant of a matrix as

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ or } \begin{vmatrix} a & b \\ c & d \end{vmatrix} \text{ or } \det A \text{ or } |A|$$

<u>Example:3.</u>

Find the determinant of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

Solution:

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\det A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = +1\times\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\times\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\times\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1(45-48) - 2(36-42) + 3(32-35) = -3 + 12 - 9 = 0$$

Example4. Find determinant of matrix of order 4x4

$$A = \begin{bmatrix} 0 & 1 & 2 & 5 \\ 2 & -1 & 2 & 3 \\ 3 & 2 & 1 & 5 \\ 1 & 0 & 4 & 0 \end{bmatrix}$$

Solution:

Two entries in 4th row are zero, so determinant is calculated by opening from 4th row.

$$\begin{aligned} \det A &= a_{41}c_{41} + a_{42}c_{42} + a_{43}c_{43} + a_{44}c_{44} \\ &= (1)c_{41} + (0)c_{42} + (4)c_{43} + (0)c_{44} \\ &= c_{41} + (4)c_{43} \end{aligned}$$

$$\det A = c_{41} + (4)c_{43} = -\begin{vmatrix} 1 & 2 & 5 \\ -1 & 2 & 3 \\ 2 & 1 & 5 \end{vmatrix} - 4\begin{vmatrix} 0 & 1 & 5 \\ 2 & -1 & 2 \\ 3 & 2 & 1 \end{vmatrix}$$

Finding values of cofactors c_{41} and c_{43}

$$\begin{aligned} \det A &= -(4) - 4(34) \\ &= -4 - 136 \\ &= -140 \end{aligned}$$

Example:5.

Solving matrix equation

Find all values of λ for which det (A) = 0 for matrix

$$A = \begin{bmatrix} \lambda - 4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda - 1 \end{bmatrix}$$

Solution: Two entries of 1st row are zero, we open it from first row

$$\det\ A = (\lambda-4)\begin{vmatrix}\lambda & 2\\ 3 & \lambda-1\end{vmatrix}$$

$$=(\lambda-4)\left[\lambda(\lambda-1)-6\right]$$

$$=(\lambda-4)\left[\lambda^2-\lambda-6\right]$$

$$=(\lambda-4)(\lambda-3)(\lambda+2)$$

We need to find the value of λ, when det A = 0

$$\Rightarrow\ (\lambda-4)(\lambda-3)(\lambda+2)=0$$

$\Rightarrow$ λ=4, λ=3 and λ= -2.

Lecture 4.3: Determinant of triangular matrices

Upper triangular matrix

In upper triangular matrix all the entries below the diagonal are zero.

$$A=\begin{bmatrix}1 & 3 & 2\\ 0 & 4 & 2\\ 0 & 0 & 5\end{bmatrix}$$

Lower triangular matrix

In lower triangular matrix all the entries above the diagonal are zero.

$$B=\begin{bmatrix}1 & 0 & 0\\ 8 & 4 & 0\\ 4 & 7 & 3\end{bmatrix}$$

Note: Determinant of triangular matrix is product of diagonal elements.

Det A = (1)(4)(5) = 20

Det B = (1)(4)(3) = 12

Example:6. The determinant of Triangular matrix

$$A = \begin{bmatrix} 2 & 4 & 5 & 3 \\ 0 & 5 & 3 & -1 \\ 0 & 0 & 3 & 9 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$\det A = (2)(4)(5)(3) = 120$$

Diagonal Matrices

Diagonal matrix is matrix whose off diagonal elements are zero.

Example:7. Determinant of Diagonal matrix

Find determinant of matrix $B = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Solution:

$$\det B = (5)(4)(3) = 60$$

Example:8.

Evaluate $\det C = \begin{vmatrix} 3 & 0 & 0 & 0 & 0 \\ -4 & 2 & 0 & 0 & 0 \\ 67 & e & 4 & 0 & 0 \\ 0 & 1 & -47 & 2 & 0 \\ \pi & -3 & 6 & -\sqrt{2} & -1 \end{vmatrix}$

Matrix C is lower triangular $\Rightarrow$ $\det C = 3 \times 2 \times 4 \times 2 \times (-1) = -48$

Example:9.

Evaluate $\det D = \begin{vmatrix} 2 & -1 & 1 & 1 \\ -3 & 2 & -4 & -3 \\ 4 & 2 & 7 & 4 \\ 2 & 3 & 11 & 2 \end{vmatrix}$

Columns 1 and 4 of matrix D are identical $\Rightarrow$ $\det D = 0$.

Lecture 5.1 Properties of Determinant

Property 1: If one row of a matrix consists entirely of zeros, then the determinant is zero.

Example1: $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 2 & 4 & 8 \end{bmatrix}, \quad \det A = 0,$

Property 2: If two rows of a matrix are identical, the determinant is zero.

Example2: $A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 2 & 4 & 8 \end{bmatrix}, \quad \det A = 0$, Row 1 and Row 2 are identical

Property 3: If in a square matrix A two rows proportional, then det A = 0**.**

Example3: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 4 & 6 \end{bmatrix}, \quad \det(A) = 0.$

Row 1 and Row 3 are proportional, as $R_3 = 2\ R_1$

Property 3: $\det(\mathbf{A}) = \det(\mathbf{A}^T)$.

Example4:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 4 & 8 \end{bmatrix}, \quad \det A = 2, \qquad A^t = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 4 \\ 3 & 2 & 8 \end{bmatrix}, \quad \det(A^t) = 2$$

Property 4: For an $n \times n$ matrix **A** and any scalar λ, $\det(\lambda \mathbf{A}) = \lambda^n \det(\mathbf{A})$.

Note:

When we multiply a matrix with a number, each entry of matrix is multiplied with the same number.

When we take common from determinant, it is taken from each row or each column.

Example5:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 4 & 8 \end{bmatrix}, \quad 4A = \begin{bmatrix} 4 & 8 & 9 \\ 0 & 4 & 8 \\ 8 & 16 & 32 \end{bmatrix},$$

$$\det(4A) = \begin{vmatrix} 4 & 8 & 9 \\ 0 & 4 & 8 \\ 8 & 16 & 32 \end{vmatrix} = (4)(4)(4)\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 4 & 8 \end{vmatrix} = 4^3\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 4 & 8 \end{vmatrix} = 64\text{x}2 = 128$$

Property 5: If A and B are of the same order, then $\det(A+B) \neq \det(A) + \det(B)$.

Property 6: If A and B are of the same order, then $\det(AB)=\det(A)\det(B)$.

Property 7: $\det(\mathbf{A}^{-1}) = \frac{1}{\det(A)}$.

Property 8: If $\det(\mathbf{A}) = 0$, then matrix A is singular matrix

Property 9: Homogeneous system of linear equations AX = 0, will have non- trivial solution if and only if detA = 0 .

Example6. Given that

$$\textbf{Let } A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \textbf{ and } \det(A) = -7, \text{find}$$

$$\text{(a)}\det(3A), \quad \text{(b)}\det(2A)^{-1}, \quad \text{(c)}\det(2A^{-1}), \ \text{(d)} \ \det\ A^T = \det A, \quad \text{(e)}\begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix}$$

Solution:

(a) $\det(3A) = 3^3 \det A = (27)(-7) = -189$

(b) $\det(2A)^{-1} = \frac{1}{\det(2A)} = \frac{1}{2^3 detA} = \frac{1}{(8)(-7)} = -\frac{1}{56}$

(c) $\det(2A^{-1}) = 2^3 \det(A^{-1}) = \frac{8}{\det A} = \frac{8}{-7} = -\frac{8}{7}$

(d) $\det(A^T)=\det(A) = -7$

(e) $$\begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix} = \underbrace{\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix}}_{\text{Taking Transpose}} = \underbrace{-\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}_{\text{Interchanging } R_2 \text{ and } R_3} = -(-7) = 7$$

Example7. Use row reduction to show that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-c)(c-a)(c-b)$$

Solution:

Using property that det A^t = detA

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \underbrace{\begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}}_{\boxed{R_2-R_1, R_3-R_1}} = \underbrace{(b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}}_{\text{taking common from } R_2 \text{ and } R_3}$$

$$=\underbrace{(b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{vmatrix}}_{\boxed{R_3-R_2}}$$ **It is triangular matrix**

$$=(b-a)(c-a)(c-b)$$

Example 8. Using properties of determinants show that

$$\begin{vmatrix} b+c & c+a & a+b \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Solution:

$$\begin{vmatrix} b+c & c+a & a+b \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} \quad \boxed{R_1+R_2}$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} \quad \text{Taking common from } R_1$$

$$= 0. \quad R_1 \text{ and } R_3 \text{ are equal}$$

Lecture 5.2 Elementary Row operations and Determinant

Let A and B be square matrices

1. *If B is obtained by interchanging two rows of A,*
   then det B = - det A
2. *If B is obtained by multiplying row of A by a nonzero constant k,*
   then det B = k det A
3. *If B is obtained from A by adding a multiple of a row A to another row of A, then det B = det A*

Example 9.

Find determinant of Matrix by using elementary row operations

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 4 & 8 \end{bmatrix}$$

Solution:

Reducing to triangular matrix, multiply row 1 by (-2) and add to row 3

$$\det A = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{vmatrix} = (1)(1)(2) = 2$$

Example10.

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 4 & 8 \end{bmatrix}$ and det A = 2. Find determinant of matrix

$$\text{(i)}\ A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 8 \\ 0 & 1 & 2 \end{bmatrix},\ \text{(ii)}\ A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix},\ \text{(iii)}\ A_3 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

Solution:

(i) The matrix A_1 can be obtained by interchanging Row 2 and Row 3 of matrix A

det A_1 = - det A = -2

(ii) The matrix A_2 can be obtained by multiplying Row 3 of matrix A by 1/2

det A_2 = ½ det A = ½ (2) = 1

(iii) The matrix A_3 can be obtained by Row operation on matrix A (-2 Row 1 to Row 3)

det A_3 = det A = 2

Example11. Given that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 6,\ \text{find(a)} \begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix}, \text{(b)} \begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix}, \text{(c)} \begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix}$$

Solution:

$$\text{(a)} \begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix} = - \underset{R_1 \leftrightarrow R_3}{\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix}} = (-1)(-1) \underset{R_2 \leftrightarrow R_3}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} = (-1)(-1)(6) = 6$$

$$\text{(b)} \begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix} = (3)(-1)(4) \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (-12)(6) = -72$$

Taking common from each row

R_1-R_3

$$\text{(c)} \begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 6$$

Lecture 5.3: Evaluation of Determinant

Finding determinant by using Properties of determinant

Example 12. Given that

$$\textbf{Let } A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \textbf{ and } \det(A) = -7, \text{find}$$

$$\text{(a)}\det(3A),\quad \text{(b)}\det(2A)^{-1},\quad \text{(c)}\det(2A^{-1}),\quad \text{(d)}\ \det\ A^{T} = \det A,\quad \text{(e)}\begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix}$$

Solution:

(a) $\det(3A) = 3^3 \det A = (27)(-7) = -189$

(b) $\det(2A)^{-1} = \frac{1}{\det(2A)} = \frac{1}{2^3 detA} = \frac{1}{(8)(-7)} = -\frac{1}{56}$

(c) $\det(2A^{-1}) = 2^3 \det(A^{-1}) = \frac{8}{\det A} = \frac{8}{-7} = -\frac{8}{7}$

(d) $\det(A^T)=\det(A) = -7$

(e) $$\begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix} = \underbrace{\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix}}_{\text{Taking Transpose}} = \underbrace{-\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}_{\text{Interchanging } R_2 \text{ and } R_3} = -(-7) = 7$$

Finding determinant by using Elementary row operations

Example12.

Use row reduction to show that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-c)(c-a)(c-b)$$

Solution:

Using property that det A^t = detA

$$\begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix} = \begin{vmatrix}1 & a & a^2\\ 0 & b-a & b^2-a^2\\ 0 & c-a & c^2-a^2\end{vmatrix} = (b-a)(c-a)\begin{vmatrix}1 & a & a^2\\ 0 & 1 & b+a\\ 0 & 1 & c+a\end{vmatrix}$$

$\boxed{R_2 - R_1, R_3 - R_1}$ **taking common from R_2 and R_3**

$$=(b-a)(c-a)\begin{vmatrix}1 & a & a^2\\ 0 & 1 & b+a\\ 0 & 0 & c-b\end{vmatrix}$$ **It is triangular matrix**

$\boxed{R_3 - R_2}$

$$=(b-a)(c-a)(c-b)$$

Finding determinant by using Properties of determinant

Example13.

Using properties of determinants show that

$$\begin{vmatrix}b+c & c+a & a+b\\ a & b & c\\ 1 & 1 & 1\end{vmatrix} = 0$$

Solution:

$$\begin{vmatrix}b+c & c+a & a+b\\ a & b & c\\ 1 & 1 & 1\end{vmatrix} = \begin{vmatrix}a+b+c & a+b+c & a+b+c\\ a & b & c\\ 1 & 1 & 1\end{vmatrix}$$ $\boxed{R_1+R_2}$

$$= (a+b+c)\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ 1 & 1 & 1\end{vmatrix}$$ Taking common from R_1

$$= 0.$$ R_1 and R_3 are equal

Example14.

Find the values of x for which the matrix does not have inverse

$$A = \begin{bmatrix}x+2 & 2x+3 & 3x+4\\ 2x+3 & 3x+4 & 4x+5\\ 3x+5 & 5x+8 & 10x+17\end{bmatrix}$$

Solution:

$$\det A = \begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 3x+5 & 5X+8 & 10x+17 \end{vmatrix}$$

By the row operations $R_3 - R_2$**, and** $R_3 - R_1$

$$= \begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ x+1 & x+1 & x+1 \\ x+2 & 2x+4 & 6x+12 \end{vmatrix}$$

Taking common $x+1$ **from Row 1 and** $x+2$ **from Row 2**

$$= (x+1)(x+2)\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 1 & 1 & 1 \\ 1 & 2 & 6 \end{vmatrix}$$

By subtracting column 1 from column 2 and column 3

$$= (x+1)(x+2)\begin{vmatrix} x+2 & x+1 & 2x+2 \\ 1 & 0 & 0 \\ 1 & 1 & 5 \end{vmatrix}$$

Opening from Row2

$$=(x+1)(x+2)(-3(x+1))$$

$$\det A = 0 \Rightarrow -3(x+1)(x+2)(x+1) = 0$$

is zero $\Rightarrow x = -1 \text{ or } x = -2$

Note: We can apply the operation in columns we perform operations on rows.

<u>Example 15</u>.

Use determinants to find which real value(s) of c make this matrix invertible:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & c \\ 2 & c & 1 \end{bmatrix}$$

Solution:

$$A = \begin{vmatrix} 1 & 2 & -1 \\ 0 & -1 & c \\ 2 & c & 1 \end{vmatrix} = 0 + (-1)\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} - c\begin{vmatrix} 1 & 2 \\ 2 & c \end{vmatrix}$$

$= -(1+2) - c(c-4) = -(c^2 - 4c + 3) = -(c-1)(c-3)$

$\det A = 0 \quad \Rightarrow \quad c = 1$ or $c = 3$

Therefore the matrix is invertible for all real values of c except $c = 1$ or $c = 3$.

Finding determinant by using Elementary row operations, reducing it to upper triangular matrix form

Example 16. Evaluate

$$\det A = \begin{vmatrix} 1 & -1 & 5 & 5 \\ 3 & 1 & 2 & 4 \\ -1 & -3 & 8 & 0 \\ 1 & 1 & 2 & -1 \end{vmatrix}.$$

Solution: Use elementary row operations to carry the matrix to upper triangular form:

$$\begin{vmatrix} 1 & -1 & 5 & 5 \\ 3 & 1 & 2 & 4 \\ -1 & -3 & 8 & 0 \\ 1 & 1 & 2 & -1 \end{vmatrix} \xrightarrow[\substack{R_3 + R_1 \\ R_4 - R_1}]{R_2 - 3R_1} \begin{vmatrix} 1 & -1 & 5 & 5 \\ 0 & 4 & -13 & -11 \\ 0 & -4 & 13 & 5 \\ 0 & 2 & -3 & -6 \end{vmatrix}$$

$$\xrightarrow[\substack{R_3 + R_2 \\ R_4 - \frac{1}{2}R_2}]{} \begin{vmatrix} 1 & -1 & 5 & 5 \\ 0 & 4 & -13 & -11 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & \frac{7}{2} & -\frac{1}{2} \end{vmatrix} \xrightarrow[R_3 \leftrightarrow R_4]{} -\begin{vmatrix} 1 & -1 & 5 & 5 \\ 0 & 4 & -13 & -11 \\ 0 & 0 & \frac{7}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & -6 \end{vmatrix}$$

$$\Rightarrow \quad \det A = -1 \times 4 \times \tfrac{7}{2} \times (-6) = +84.$$

Lecture 6.1 Applications of Determinants

Minors and cofactors of a Matrix

$$\text{Let A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Definition 1:

Given a matrix A, the Minor of $a_{ij} \equiv M_{ij}$, is determinant obtained from A by removing i^{th} row and j^{th} column.

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$ **is determinant obtained by deleting 1st row and 1st column**

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \quad M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \quad M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, \quad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, \quad M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}, \quad M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \quad M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$\boxed{\textbf{Cofactor of } a_{ij} = C_{ij} = (-1)^{i+j} M_{ij}}$$

Signs of Cofactors

For 2x2 – matrix $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$

For 3x3 – matrix $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

For 4x4 – matrix $\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$

Definition 2:

Given a matrix **A**, the **cofactor** of the element $\mathbf{a_{ij}}$ is a scalar obtained by multiplying together the term $\mathbf{(-1)^{i+j}}$ and the minor obtained from **A** by removing the i^{th} row and the j^{th} column.

Example:1.

Find all minors and cofactors of the matrix

$$A=\begin{bmatrix}3 & 4 & -1\\ 1 & 0 & 3\\ 2 & 5 & -4\end{bmatrix}$$

Solution:

$$\mathrm{M}_{11}=\begin{vmatrix}0 & 3\\ 5 & -4\end{vmatrix}=-15,\quad \mathrm{M}_{12}=\begin{vmatrix}1 & 3\\ 2 & -4\end{vmatrix}=-10,\quad \mathrm{M}_{13}=\begin{vmatrix}1 & 0\\ 2 & -5\end{vmatrix}=5$$

$$\mathrm{M}_{21}=\begin{vmatrix}4 & -1\\ 5 & -4\end{vmatrix}=-11,\quad \mathrm{M}_{22}=\begin{vmatrix}3 & -1\\ 2 & -4\end{vmatrix}=-10,\quad \mathrm{M}_{23}=\begin{vmatrix}3 & 4\\ 2 & 5\end{vmatrix}=7$$

$$\mathrm{M}_{31}=\begin{vmatrix}4 & -1\\ 0 & 3\end{vmatrix}=12,\quad \mathrm{M}_{32}=\begin{vmatrix}3 & -1\\ 1 & 0\end{vmatrix}=10,\quad \mathrm{M}_{33}=\begin{vmatrix}3 & 4\\ 1 & 0\end{vmatrix}=-4$$

$$\boxed{\textbf{Cofactor of } a_{ij}=C_{ij}=(-1)^{i+j}M_{ij}}$$

$$C_{11}=-15,\quad C_{12}=10,\quad C_{13}=5$$
$$C_{21}=11,\quad C_{22}=-10,\quad C_{23}=-7$$
$$C_{31}=12,\quad C_{32}=-10,\quad C_{33}=-4$$

NOTE: Matrix of cofactors , $C=\begin{bmatrix}-15 & 10 & 5\\ 11 & -10 & -7\\ 12 & -10 & -4\end{bmatrix}$

<u>NOTE: Determinant of matrix of Cofactors by the method of cofactors</u>

$$\det(A)=a_{11}C_{11}+a_{12}C_{12}+a_{13}C_{13}$$
$$\det(A)=a_{21}C_{21}+a_{22}C_{22}+a_{23}C_{23}$$
$$\det(A)=a_{31}C_{31}+a_{32}C_{32}+a_{33}C_{33}$$

The above equations can be used to check that the cofactors are found correctly as the values of determinants found must be equal, we open matrix from any row or column.

Example: 2 .
Find the determinant of the matrix A by method of cofactors,

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}$$

Solution:
Using the cofactors found in the last example.
Expanding from First row

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$
$$= 3(-15)+4(10)+(-1)(5)$$
$$=-45 + 40 - 5 = -10$$

NOTE: 3. We can find determinant by opening matrix from second or third row or first column, the value of the determinant will be same

$$\det(A) = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$
$$=(1)(11)+0(-10)+3(-7)=11 - 21 = -10$$

$$\det(A) = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$$
$$= 2(12) +5(-10)+(-4)(-4) =24 - 50 +16 = -10$$

NOTE : 4. Determinant of A can be obtained by multiplying any row or any column of matrix A with the corresponding cofactors of the matrix.

NOTE: 5. Determinant of matrix A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

det A = $a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$.

Lecture 6.2 : Inverse by method of Cofactors

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \det \mathbf{A} \neq 0.$$

Step:1. Find Matrix of cofactors

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

Step : 2. Find Adjoint of matrix A , adj(A)

$$\mathbf{Adj(A)} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

Step: 3.

If A is an invertible matrix, det(A) $\neq$ 0, then

$$A^{-1} = \frac{1}{\det A}[adj(A)]$$

Example: 3 . Find A^{-1} of matrix A

$$A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{bmatrix}$$ by the method of cofactors.

Solution: Cofactors of the matrix A are

$$C_{11} = \begin{vmatrix} 3 & 2 \\ 0 & -4 \end{vmatrix} = -12, C_{12} = -\begin{vmatrix} 0 & 2 \\ -2 & -4 \end{vmatrix} = -4, C_{13} = \begin{vmatrix} 0 & 3 \\ -2 & 0 \end{vmatrix} = 6$$

$$C_{21} = -\begin{vmatrix} 0 & 3 \\ 0 & 4 \end{vmatrix} = 0, \quad C_{22} = \begin{vmatrix} 2 & 3 \\ -2 & -4 \end{vmatrix} = -2, \; C_{23} = -\begin{vmatrix} 2 & 0 \\ -2 & 0 \end{vmatrix} = 0,$$

$$C_{31} = \begin{vmatrix} 0 & 3 \\ 3 & 2 \end{vmatrix} = -9, \quad C_{32} = -\begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} = -4, \quad C_{33} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$$

Matrix of cofactors, $C = \begin{bmatrix} -12 & -4 & 6 \\ 0 & -2 & 0 \\ -9 & -4 & 6 \end{bmatrix}$

Adjoint of matrix A, $\text{adj}(A) = \begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix}$

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$
$$= 2(-12)+0(-4)+3(6)$$
$$=-24 +18 = - 6 \neq 0$$

Inverse of the matrix A is

$$A^{-1} = \frac{1}{\det A}[adj(A)] = \frac{1}{-6}\begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix}$$

NOTE :

If we can find A^{-1} , then solution of linear system AX = B is $X = A^{-1}B$

Lecture 6.3 : Cramer's Rule

Using determinants to solve a system of linear equations.

Theorem:

If A is $n \times n$ matrix with det(A)$\neq$0, then the linear system AX = B has a <u>unique</u> solution X = (x_j) given by

$$x_j = \frac{\det(A_j)}{\det(A)} \quad , j = 1,2,\ldots,n$$

Where A_j is the matrix obtained by replacing the jth column of A by B.

<u>NOTE:</u> If A is 3x3 matrix , then the solution of the system AX = B is

$$\mathrm{x} = \frac{\det(\mathrm{A}_1)}{\det(\mathrm{A})}, \quad \mathrm{y} = \frac{\det(\mathrm{A}_2)}{\det(\mathrm{A})}, \quad \mathrm{z} = \frac{\det(\mathrm{A}_3)}{\det(\mathrm{A})}$$

Example 4.

Use Cramer's Rule to solve

$$\begin{aligned} 4x + 5y \quad\quad &= 2 \\ 11x + y + 2z &= 3 \\ x + 5y + 2z &= 1 \end{aligned}$$

Solution:

$$\mathbf{A} = \begin{bmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{bmatrix}, \quad \mathbf{A_1} = \begin{bmatrix} 2 & 5 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & 2 \end{bmatrix}, \quad \mathbf{A_2} = \begin{bmatrix} 4 & 2 & 0 \\ 11 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}, \quad \mathbf{A_3} = \begin{bmatrix} 4 & 5 & 2 \\ 11 & 1 & 3 \\ 1 & 5 & 1 \end{bmatrix}$$

det(A) = -132, det(A_1) =-36, det(A_2) = -24, det(A_3) = 12

$$x = \frac{\det(A_1)}{\det(A)} = \frac{-36}{-132} = \frac{3}{11},$$

$$y = \frac{\det(A_2)}{\det(A)} = \frac{-24}{-132} = \frac{2}{11},$$

$$z = \frac{\det(A_{31})}{\det(A)} = \frac{12}{-132} = \frac{-1}{11}$$

NOTE: If det(A) = 0, then there does not exist any solution of the system.

Example:3. Find determinant of matrix if $A = \begin{array}{c} \begin{matrix} + & - & + & - \end{matrix} \\ \begin{bmatrix} 0 & 1 & 2 & 5 \\ 2 & -1 & 2 & 3 \\ 3 & 2 & 1 & 5 \\ 1 & 0 & 4 & 0 \end{bmatrix} \end{array}$ ← sign Rule

Solution: Expanding from 4^{th} row

$$\text{Det}(A) = -(1)\begin{vmatrix} 1 & 2 & 5 \\ -1 & 2 & 3 \\ 2 & 1 & 5 \end{vmatrix} + (0)\begin{vmatrix} 0 & 2 & 5 \\ 2 & 2 & 3 \\ 3 & 1 & 5 \end{vmatrix} - (4)\begin{vmatrix} 0 & 1 & 5 \\ 2 & -1 & 3 \\ 3 & 2 & 5 \end{vmatrix} + (0)\begin{vmatrix} 0 & 1 & 2 \\ 2 & -1 & 2 \\ 3 & 2 & 1 \end{vmatrix}$$

$$= -(1)(4) + (0)(?) - (4)(34) + (0)(?)$$

$$= -4 - 136 = -140.$$

Example : 4. Find all values of λ for which $\det(A) = 0$ for matrix ≡ which make A without inverse A^{-1}

$$A = \begin{bmatrix} \lambda - 4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda - 1 \end{bmatrix}.$$

Solution: $\det(A) = (\lambda-4)\begin{vmatrix} \lambda & 2 \\ 3 & \lambda - 1 \end{vmatrix} - (0)\begin{vmatrix} 0 & 2 \\ 0 & \lambda - 1 \end{vmatrix} + (0)\begin{vmatrix} 0 & \lambda \\ 0 & 3 \end{vmatrix}$

$= (\lambda-4)\,[\lambda(\lambda-1) - 6]$

$= (\lambda-4)\,[\lambda^2 - \lambda - 6]$

$= (\lambda-4)(\lambda - 3)(\lambda+2)$

$\det(A) = 0.$

$(\lambda-4)(\lambda - 3)(\lambda+2) = 0.$

$\Rightarrow \lambda = 4,\ \lambda = 3,\ \lambda = -2.$

How to find $|A|$ through row operations

3.4 Evaluating Determinant by row operations

1. If matrix A_1 is obtained from matrix A by the interchange of two rows, then $\boxed{\det(A_1) = -\det(A).}$

2. If matrix A_2 is obtained from matrix A by the multiplication of a row of A by a constant k, then $\boxed{\det(A_2) = k\det(A).}$

3. If matrix A_3 is obtained from the matrix A by addition of a multiple of one row to another row, then $\boxed{\det(A_3) = \det(A).}$

Example:5. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 4 & 8 \end{bmatrix}$, and det(A) = 2. Find determinant of

(i) $A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 8 \\ 0 & 1 & 2 \end{bmatrix}$, (ii) $A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$, (iii) $A_3 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

Solution: (i) A_1 is obtained from A by interchanging R_2 and R_3 of A,
$\det(A_1) = -\det(A) = -2$.

(ii) A_2 is obtained from A by multiplying R_3 of A by $\frac{1}{2}$,
$\det(A_2) = \frac{1}{2}\det(A) = \frac{1}{2}(2) = 1$.

(iii) A_3 is obtained by row operation $-2R_2+R_1$,
$\det(A_3) = \det(A) = 2$.

NOTE:

Important

1. If A is any square matrix that contains a row of zeros, then det(A) = 0.
2. If a square matrix has two proportional rows, then det(A) = 0.
3. In case of upper or lower triangular matrix, determinant is the product of the diagonal elements.

Special cases

Upper triangular matrix

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \det(A) = a_{11}a_{22}a_{33}$$

Lower triangular matrix

$$B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}, \quad \det(B) = a_{11}a_{22}a_{33}$$

Example:6.

Given that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 6$, find (a) $\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix}$, (b) $\begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix}$

(c) $\begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix}$, (d) $\begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix}$.

Solution:

(a) $$\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix} = -\underset{R_1 \leftrightarrow R_3}{\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix}} = (-1)(-1)\underset{R_2 \leftrightarrow R_3}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} = (-1)(-1)(6) = 6$$

(b) $$\begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix} = (3)(-1)(4)\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (-12)(6) = -72$$

(c) $$\begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix} = \underset{R_1 - R_3}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} = 6$$

(d) $$\begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix} = (-3)\underset{4R_1 + R_3}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} = (-3)(6) = -18$$

Important

Example:7. Evaluate the determinant by row reduction

$$\text{Det } A = \begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ -2 & -7 & 0 & -4 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}$$

Solution:

$$\det A = \begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & -1 & 2 & 6 & 8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix} \qquad 2R_1 + R_2, -2R_3 + R_4$$

$$= \begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & -1 & 2 & 6 & 8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix} \qquad -R_4 + R_5$$

$$= (1)(-1)(1)(1)(2) = -2$$

Example:8. Find the value(s) of x if det A = -12, where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & x-3 & -3 \\ 1 & x-4 & 0 \end{bmatrix}$$

Solution: Performing row operations $-2R_1+R_2$, $-R_1+R_3$

$$\det A = \begin{vmatrix} 1 & 0 & 0 \\ 0 & x-3 & -3 \\ 0 & x-4 & 0 \end{vmatrix} = (1)\begin{vmatrix} x-3 & -3 \\ x-4 & 0 \end{vmatrix} - (0) + (0)$$

$$= 3(x-4)$$

$$\det A = -12 \quad \Rightarrow -3x - 12 = -12$$

$$-3x = 0$$

$$x = 0 //$$

NOTE: Operations on columns are same as on rows.

Important

Theorem: For an nxn matrix A, following are equivalent:

1. $\det(A) \neq 0$,
2. A^{-1} exists, and
3. $AX = B$ has a unique solution for any B.
4. A is invertible

3.5 Properties of Determinantial Function

Rules

1. **If A is a nxn matric $\det(kA) = k^n \det(A)$,**
2. **$\det(A+B) \neq \det(A) + \det(B)$,**
3. **$\det(AB) = \det(A).\det(B)$,**
4. $\det(A^{-1}) = \dfrac{1}{\det A}$,
5. **A square matrix is invertible if and only if $\det(A) \neq 0$,and**
6. **$\det(A^t) = \det(A)$**

Example :9. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and $\det(A) = -7$ find

(a) $\det(3A)$, (b) $\det(2A)^{-1}$, (c) $\det(2A^{-1})$ and (d) $\begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix}$

Solution: a. $\det(3A) = 3^3 \det A = 27(-7) = -189$

b. $$\det(2A)^{-1} = \frac{1}{\det(2A)} = \frac{1}{2^3\det(A)} = \frac{1}{8(-7)} = \frac{-1}{56}$$

c. $$\det(2A^{-1}) = 2^3\det(A) = \frac{2^3}{\det(A)} = \frac{8}{-7} = \frac{-8}{7}$$

d. $$\begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix} = \begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} = -\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -(-7) = 7$$

Example:10. Use row reduction to show that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$$

Solution.

$\det(A) = \det(A^t)$

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$$

$R_2 - R_1, R_3 - R_1$

$$= (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{vmatrix}$$

$R_3 - R_2$

$$= (b-c)(c-a)(c-b)$$

Example:11. Without directly evaluating by using properties of determinant show that

$$\begin{vmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

Solution:

$$\begin{vmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} R_1 + R_2$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 0.$$

Example: 3. Find A^{-1} of matrix A

$$A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{bmatrix} \text{ by the method of cofactors.}$$

Solution: Cofactors of the matrix A are

$$C_{11} = \begin{vmatrix} 3 & 2 \\ 0 & -4 \end{vmatrix} = -12, C_{12} = -\begin{vmatrix} 0 & 2 \\ -2 & -4 \end{vmatrix} = -4, C_{13} = \begin{vmatrix} 0 & 3 \\ -2 & 0 \end{vmatrix} = 6$$

$$C_{21} = -\begin{vmatrix} 0 & 3 \\ 0 & 4 \end{vmatrix} = 0, \quad C_{22} = \begin{vmatrix} 2 & 3 \\ -2 & -4 \end{vmatrix} = -2, \; C_{23} = -\begin{vmatrix} 2 & 0 \\ -2 & 0 \end{vmatrix} = 0,$$

$$C_{31} = \begin{vmatrix} 0 & 3 \\ 3 & 2 \end{vmatrix} = -9, \quad C_{32} = -\begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} = -4, \quad C_{33} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$$

$$\text{Matrix of cofactors,} \quad C = \begin{bmatrix} -12 & -4 & 6 \\ 0 & -2 & 0 \\ -9 & -4 & 6 \end{bmatrix}$$

$$\text{Adjoint of matrix A,} \; \text{adj(A)} = \begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= 2(-12)+0(-4)+3(6) \\ &= -24 + 18 = -6 \neq 0 \end{aligned}$$

Inverse of matrix A is

$$A^{-1} = \frac{1}{\det A}[adj(A)] = \frac{1}{-6}\begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix}$$

NOTE :

> If we can find A^{-1}, then solution of linear system $AX = B$ is $\quad X = A^{-1}B$

3.7 Cramer's Rule

If A is nxn marix with $\det(A) \neq 0$, then the linear system $AX = B$ has a unique solution $X = (x_j)$ given by

$$x_j = \frac{\det(A_j)}{\det(A)} \quad , j = 1, 2, \ldots, n$$

Where A_j is the matrix obtained by replacing the jth column of A by B.

NOTE: If A is 3x3 matrix , then the solution of the system $AX = B$ is

$$x = \frac{\det(A_1)}{\det(A)}, \quad y = \frac{\det(A_2)}{\det(A)}, \quad z = \frac{\det(A_{31})}{\det(A)}$$

Example: Use Cramer's Rule to solve

$$\begin{aligned} 4x + 5y \quad &= 2 \\ 11x + y + 2z &= 3 \\ x + 5y + 2z &= 1 \end{aligned}$$

Solution: $A = \begin{bmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{bmatrix}$, $A_1 = \begin{bmatrix} 2 & 5 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & 2 \end{bmatrix}$; $A_2 = \begin{bmatrix} 4 & 2 & 0 \\ 11 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}$, $A_3 = \begin{bmatrix} 4 & 5 & 2 \\ 11 & 1 & 3 \\ 1 & 5 & 1 \end{bmatrix}$

$\det(A) = -132$, $\det(A_1) = -36$, $\det(A_2) = -24$, $\det(A_3) = 12$

★

$$x = \frac{\det(A_1)}{\det(A)} = \frac{-36}{-132} = \frac{3}{11},$$

$$y = \frac{\det(A_2)}{\det(A)} = \frac{-24}{-132} = \frac{2}{11},$$

$$z = \frac{\det(A_{31})}{\det(A)} = \frac{12}{-132} = \frac{-1}{11}$$

★★ NOTE: when $\det(A) = 0$, then there does not exist any solution of the system. (If the system is not homogenous)

TIME: 90 min
M - 107

KING SAUD UNIVERSITY

FULL MARKS: 50

DEPARTMENT OF MATHEMATICS

(SEMESTER I, 1436 -1437) FIRST MID-TERM

Question:1 . Let

$$\begin{aligned} x - y - z &= 0 \\ 2x + y + z &= 3 \\ x + 2y + z &= 0 \end{aligned}$$

(a) Write the above system of linear equations in the form $AX=B$, [12]
(b) Find A^{-1} ,if exists ,by using elementary matrix method, and
(c) Use A^{-1} to solve the above system of equations.

Question: 2 . (a)Evaluate det(A) by using row reduction, where

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 2 & 0 & -1 & 3 \\ 0 & 2 & 1 & 4 \\ -2 & -1 & 0 & 1 \end{bmatrix}$$ [7]

(b) Find all values of x for which matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & x^2 - 2 \end{bmatrix}$$ is invertible. [7]

Question: 3 . Solve the linear system by using Crammer's Rule

$$\begin{aligned} 3x_1 + 5x_2 &= 7 \\ 6x_1 + 2x_2 + 4x_3 &= 10 \\ -x_1 + 4x_2 - 3x_3 &= 0 \end{aligned}$$ [12]

Question: 4 . Suppose the points $(1,1),(2,3)$ and $(3,4)$ lie on the curve

$$y = ax^2 + bx + c.$$

i. Find the system of linear equations in a, b and c.
ii. Solve the system by Gauss – Jordon method to find a, b and c.
iii. Write the equation of the curve. [12]

2015/10/14

TIME: 90 min	**KING SAUD UNIVERSITY**	**FULL MARKS: 50**
M - 107	**DEPARTMENT OF MATHEMATICS**	
	(SEMESTER I, 1436 -1437) FIRST MID-TERM	

Question:1 . **Let**

$$\begin{aligned} x - y - z &= 0 \\ 2x + y + z &= 3 \\ x + 2y + z &= 0 \end{aligned}$$

(a) Write the above system of linear equations in the form $AX=B$, **[12]**
(b) Find A^{-1} **,if exists ,by using elementary matrix method, and**
(c) Use A^{-1} **to solve the above system of equations.**

Question: 2 . (a)Evaluate det(A) by using row reduction, where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 2 & 0 & -1 & 3 \\ 0 & 2 & 1 & 4 \\ -2 & -1 & 0 & 1 \end{bmatrix}$$ **[7]**

(b) Find all values of x **for which matrix**

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & x^2 - 2 \end{bmatrix}$$ **is invertible.** **[7]**

Question: 3 . Solve the linear system by using Crammer's Rule

$$\begin{aligned} 3x_1 + 5x_2 \qquad &= 7 \\ 6x_1 + 2x_2 + 4x_3 &= 10 \\ -x_1 + 4x_2 - 3x_3 &= 0 \end{aligned}$$ **[12]**

Question: 4 . Suppose the points $(1,1),(2,3)$ and $(3,4)$ **lie on the curve**

$$y = ax^2 + bx + c.$$

i. **Find the system of linear equations in a, b and c.**
ii. **Solve the system by Gauss – Jordon method to find a, b and c.**
iii. **Write the equation of the curve.** **[12]**