⑧ Solve each of the following system using Gauss-Gordan elimination method

(a)
$$2x_1 - 3x_2 = -2$$
$$2x_1 + x_2 = 1$$
$$3x_1 + 2x_2 = 1$$

(b)
$$4x_1 - 8x_2 = 12$$
$$3x_1 - 6x_2 = 9$$
$$-2x_1 + 4x_2 = -6$$

1-3 : ③ Solve the system by any method

(a)
$$2x - y - 3z = 0$$
$$-x + 2y - 3z = 0$$
$$x + y + 4z = 0$$

1.4 : ④ (g) Let $E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}$

Find $3E^t - 3D^t$

1.5 : ⑦ Let A be invertible matrix such that $A^{-1} = \begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}$, find A.

1.6 : ⑤ (b) Find A^{-1} where $A = \begin{pmatrix} -3 & 6 \\ 4 & 5 \end{pmatrix}$ by elementary row operations.

⑦ (a) Find A^{-1}, where $A = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{bmatrix}$ by elementary row operations.

1.7 : ③ By $A^{-1}b$, Solve the system:
$$x_1 + 3x_2 + x_3 = 4$$
$$2x_1 + 2x_2 + x_3 = -1$$
$$2x_1 + 3x_2 + x_3 = 3$$

⑱ Find conditions on b's must satisfy for the system to be consistent
$$x_1 - 2x_2 - x_3 = b_1$$
$$-4x_1 + 5x_2 + 2x_3 = b_2$$
$$-4x_1 + 7x_2 + 4x_3 = b_3$$

1) Evaluate $\begin{vmatrix} -2 & 7 & 6 \\ 5 & 1 & -2 \\ 3 & 8 & 4 \end{vmatrix}$

---

2) Solve for x : $\begin{vmatrix} x & -1 \\ 3 & 1-x \end{vmatrix} = \begin{vmatrix} 1 & 0 & -3 \\ 2 & x & -6 \\ 1 & 3 & x-5 \end{vmatrix}$

---

3) Find $\det \begin{bmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{bmatrix}$

---

4) Given $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6$, find

(i) $\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix}$, (ii) $\begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix}$

---

5) Without directly evaluating, show that:

$$\begin{vmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0$$

---

6) Find A^{-1}, by using $\operatorname{adj} A$, $A = \begin{bmatrix} 2 & 5 & 5 \\ -1 & -1 & 0 \\ 2 & 4 & 3 \end{bmatrix}$

---

7) Solve by Crammer's rule, where it applies

$$x - 4y + z = 4$$
$$4x - y + 2z = -1$$
$$2x + 2y - 3z = -20$$

EXERCISES 10.3

Exer. 1–10: Given $\mathbf{a} = \langle -2, 3, 1\rangle$, $\mathbf{b} = \langle 7, 4, 5\rangle$, and $\mathbf{c} = \langle 1, -5, 2\rangle$, find the number.

1 $\mathbf{a}\cdot\mathbf{b}$

2 $\mathbf{b}\cdot\mathbf{c}$

3 (a) $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})$ (b) $\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$

4 (a) $(\mathbf{a}-\mathbf{c})\cdot\mathbf{b}$ (b) $\mathbf{a}\cdot\mathbf{b}-\mathbf{c}\cdot\mathbf{b}$

5 $(2\mathbf{a}+\mathbf{b})\cdot 3\mathbf{c}$

6 $(\mathbf{a}-\mathbf{b})\cdot(\mathbf{b}+\mathbf{c})$

7 $\text{comp}_{\mathbf{c}}\,\mathbf{b}$

8 $\text{comp}_{\mathbf{b}}\,\mathbf{c}$

9 $\text{comp}_{\mathbf{b}}\,(\mathbf{a}+\mathbf{c})$

10 $\text{comp}_{\mathbf{c}}\,\mathbf{c}$

Exer. 11–14: Find the angle between a and b.

11 $\mathbf{a} = -4\mathbf{i}+8\mathbf{j}-3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i}+\mathbf{j}+\mathbf{k}$

12 $\mathbf{a} = \mathbf{i}-7\mathbf{j}+4\mathbf{k}$, $\mathbf{b} = 5\mathbf{i}-\mathbf{k}$

13 $\mathbf{a} = \langle -2, -3, 0\rangle$, $\mathbf{b} = \langle -6, 0, 4\rangle$

14 $\mathbf{a} = \langle 3, -5, -1\rangle$, $\mathbf{b} = \langle 2, 1, -3\rangle$

Exer. 15–16: Show that a and b are orthogonal.

15 $\mathbf{a} = 3\mathbf{i}-2\mathbf{j}+\mathbf{k}$, $\mathbf{b} = 4\mathbf{i}+5\mathbf{j}-2\mathbf{k}$

16 $\mathbf{a} = \langle 4, -1, -2\rangle$, $\mathbf{b} = \langle 2, -2, 5\rangle$

Exer. 17–18: Find all values of c such that a and b are orthogonal.

17 $\mathbf{a} = \langle c, -2, 3\rangle$, $\mathbf{b} = \langle c, c, -5\rangle$

18 $\mathbf{a} = 4\mathbf{i}+2\mathbf{j}+c\mathbf{k}$, $\mathbf{b} = \mathbf{i}+22\mathbf{j}-3c\mathbf{k}$

Exer. 19–24: Given points $P(3, -2, -1)$, $Q(1, 5, 4)$, $R(2, 0, -6)$, and $S(-4, 1, 5)$, find the indicated quantity.

19 $\overrightarrow{PQ}\cdot\overrightarrow{RS}$

20 $\overrightarrow{QS}\cdot\overrightarrow{RP}$

21 The angle between $\overrightarrow{PQ}$ and $\overrightarrow{RS}$

22 The angle between $\overrightarrow{QS}$ and $\overrightarrow{RP}$

23 The component of $\overrightarrow{PS}$ along $\overrightarrow{QR}$

24 The component of $\overrightarrow{QR}$ along $\overrightarrow{PS}$

Exer. 25–26: If the vector a represents a constant force, find the work done when its point of application moves along the line segment from P to Q.

25 $\mathbf{a} = -\mathbf{i}+5\mathbf{j}-3\mathbf{k}$; $P(4, 0, -7)$, $Q(2, 4, 0)$

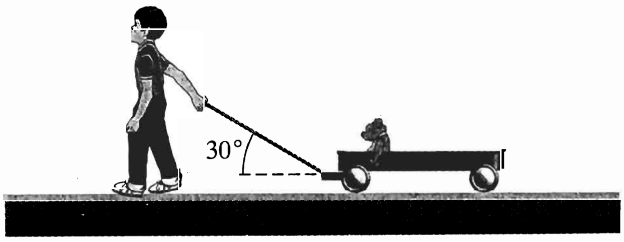
26 $\mathbf{a} = \langle 8, 0, -4\rangle$; $P(-1, 2, 5)$, $Q(4, 1, 0)$

27 A constant force of magnitude 4 lb has the same direction as the vector $\mathbf{a} = \mathbf{i}+\mathbf{j}+\mathbf{k}$. If distance is measured in feet, find the work done if the point of application moves along the y-axis from $(0, 2, 0)$ to $(0, -1, 0)$.

28 A constant force of magnitude 5 N (Newtons) has the same direction as the positive z-axis. If distance is measured in meters, find the work done if the point of application moves along a line from the origin to the point $P(1, 2, 3)$.

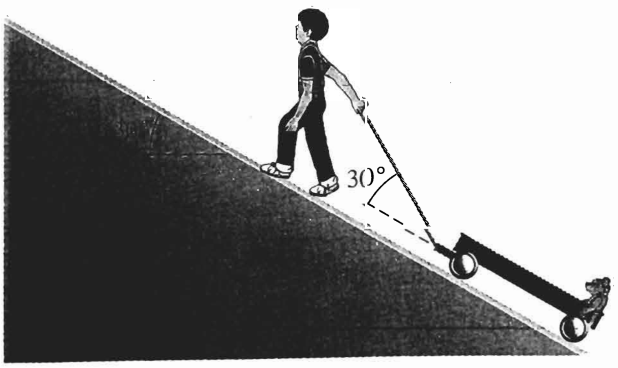
29 A child pulls a wagon along level ground by exerting a force of 20 lb on a handle that makes an angle of 30° with the horizontal (see figure). Find the work done in pulling the wagon 100 ft.

Exercise 29



30 Refer to Exercise 29. Find the work done if the wagon is pulled, with the same force, 100 ft up an incline that makes an angle of 30° with the horizontal (see figure).

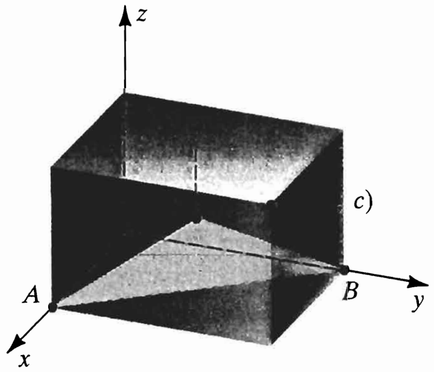
Exercise 30



31 If AB is a diameter of a sphere with center O and radius r and if P is a third point on the sphere, use vectors to show that APB is a right triangle. (*Hint:* Let $\mathbf{v}_1 = \overrightarrow{OA}$ and $\mathbf{v}_2 = \overrightarrow{OP}$, and write $\overrightarrow{PA}$ and $\overrightarrow{PB}$ in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$.)

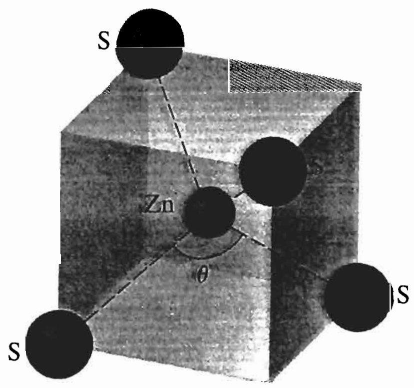
32 A rectangular box has length a, width b, and height c (see figure). If P is the center of the box, use vectors to find an expression for angle APB in terms of a, b, and c.

Exercise 32



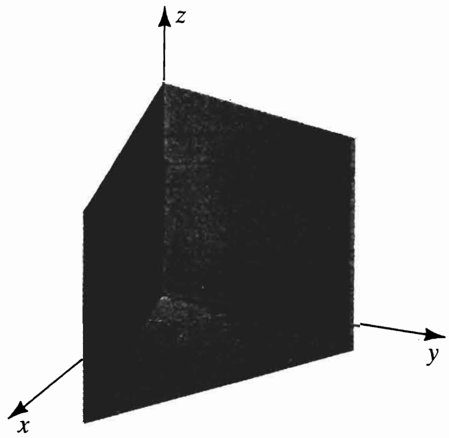
33 Refer to Exercise 32. In the mineral sphalerite, each zinc atom is surrounded by four sulphur atoms, which form a tetrahedron with the zinc atom at its center (see figure). The *bond angle* θ is the angle formed by the S–Zn–S combination. Use vectors to show that the *tetrahedral angle* θ is approximately 109.5°.

Exercise 33



34 Given a sequence A–B–C–D of four bonded atoms, the angle between the plane formed by A, B, and C and the plane formed by B, C, and D is called the *torsion angle* θ of the bond. This torsion angle is used to explain the stability of molecular structures. If segment BC is placed along the z-axis (see figure), how can θ be computed in terms of the components of vectors $\overrightarrow{BA}$ and $\overrightarrow{CD}$?

Exercise 34



35 The *direction angles* of a nonzero vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ are defined as the angles α, β, and γ between the vectors **i**, **j**, and **k**, respectively, and the vector **a**. The *direction cosines* of **a** are $\cos\alpha$, $\cos\beta$, and $\cos\gamma$. Prove the following:

(a) $\cos\alpha = \dfrac{a_1}{\|\mathbf{a}\|}, \quad \cos\beta = \dfrac{a_2}{\|\mathbf{a}\|}, \quad \cos\gamma = \dfrac{a_3}{\|\mathbf{a}\|}$

(b) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

36 Refer to Exercise 35.

(a) Find the direction cosines of $\mathbf{a} = \langle -2, 1, 5 \rangle$.

(b) Find the direction angles and the direction cosines of **i, j,** and **k**.

(c) Find two unit vectors that satisfy the condition

$$\cos\alpha = \cos\beta = \cos\gamma.$$

37 Three nonzero numbers l, m, and n are *direction numbers* of a nonzero vector **a** if they are proportional to the direction cosines—that is, if there exists a positive number k such that

$$l = k\cos\alpha, \qquad m = k\cos\beta, \qquad n = k\cos\gamma.$$

If $d = (l^2 + m^2 + n^2)^{1/2}$, prove that

$$\cos\alpha = l/d, \qquad \cos\beta = m/d, \qquad \cos\gamma = n/d.$$

38 Refer to Exercise 37. If l_1, m_1, n_1 and l_2, m_2, n_2 are direction numbers of **a** and **b**, respectively, prove that

(a) **a** and **b** are orthogonal if and only if

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

(b) **a** and **b** are parallel if and only if there is a number k such that $l_1 = kl_2$, $m_1 = km_2$, and $n_1 = kn_2$

EXERCISES 10.4

Exer. 1–10: Find a × b.

1 $\mathbf{a} = \langle 1, -2, 3 \rangle$, $\mathbf{b} = \langle 2, 1, -4 \rangle$

2 $\mathbf{a} = \langle -5, 1, -1 \rangle$, $\mathbf{b} = \langle 3, 6, -2 \rangle$

3 $\mathbf{a} = \langle 0, 1, 2 \rangle$, $\mathbf{b} = \langle 1, 2, 0 \rangle$

4 $\mathbf{a} = \langle 0, 0, 4 \rangle$, $\mathbf{b} = \langle -7, 1, 0 \rangle$

5 $\mathbf{a} = 5\mathbf{i} - 6\mathbf{j} - \mathbf{k}$, $\mathbf{b} = 3\mathbf{i} + \mathbf{k}$

6 $\mathbf{a} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{b} = -5\mathbf{j} + 2\mathbf{k}$

7 $\mathbf{a} = -3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 9\mathbf{i} - 3\mathbf{j} - 6\mathbf{k}$

8 $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 8\mathbf{k}$, $\mathbf{b} = 5\mathbf{j}$

9 $\mathbf{a} = 4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

10 $\mathbf{a} = 3\mathbf{i}$, $\mathbf{b} = 4\mathbf{k}$

Exer. 11–12: Use the vector product to show that a and b are parallel.

11 $\mathbf{a} = \langle -6, -10, 4 \rangle$, $\mathbf{b} = \langle 3, 5, -2 \rangle$

12 $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$, $\mathbf{b} = -6\mathbf{i} + 3\mathbf{j} - 12\mathbf{k}$

Exer. 13–14: Let $\mathbf{a} = \langle 2, 0, -1 \rangle$, $\mathbf{b} = \langle -3, 1, 0 \rangle$, and $\mathbf{c} = \langle 1, -2, 4 \rangle$.

13 Find $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ and $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

14 Find $\mathbf{a} \times (\mathbf{b} - \mathbf{c})$ and $(\mathbf{a} \times \mathbf{b}) - (\mathbf{a} \times \mathbf{c})$.

Exer. 15–18: (a) Find a vector perpendicular to the plane determined by P, Q, and R. (b) Find the area of the triangle PQR.

15 $P(1, -1, 2)$, $Q(0, 3, -1)$, $R(3, -4, 1)$

16 $P(-3, 0, 5)$, $Q(2, -1, -3)$, $R(4, 1, -1)$

17 $P(4, 0, 0)$, $Q(0, 5, 0)$, $R(0, 0, 2)$

18 $P(-1, 2, 0)$, $Q(0, 2, -3)$, $R(5, 0, 1)$

Exer. 19–20: Refer to Example 3. Find the distance from P to the line through Q and R.

19 $P(3, 1, -2)$, $Q(2, 5, 1)$, $R(-1, 4, 2)$

20 $P(-2, 5, 1)$, $Q(3, -1, 4)$, $R(1, 6, -3)$

21 If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$, prove that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Exer. 22–23: Use Example 4 and Exercise 21 to find the volume of the box having adjacent sides AB, AC, and AD.

22 $A(0, 0, 0)$, $B(1, -1, 2)$, $C(0, 3, -1)$, $D(3, -4, 1)$

23 $A(2, 1, -1)$, $B(3, 0, 2)$, $C(4, -2, 1)$, $D(5, -3, 0)$

24 If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are represented by vectors with a common initial point, show that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$ if and only if the vectors are coplanar.

25 Prove that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$ for all vectors $\mathbf{a}$ and $\mathbf{b}$.

26 If $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$ and $\mathbf{a} \neq \mathbf{0}$, does it follow that $\mathbf{b} = \mathbf{c}$? Explain.

27 Let $\mathbf{a} \neq \mathbf{0}$. If $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$ *and* $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, prove that $\mathbf{b} = \mathbf{c}$.

Exer. 28–31: Prove the given property if $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$, and m is a scalar.

28 $(m\mathbf{a}) \times \mathbf{b} = m(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (m\mathbf{b})$

29 $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c})$

30 $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$

31 $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

Exer. 32–37: Verify *without using components* for the vectors.

32 $(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b}) = 2(\mathbf{b} \times \mathbf{a})$

33 $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$
(*Hint:* Use (vi) of Theorem (10.33).)

34 $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$

35 $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} \mathbf{a} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{c} \\ \mathbf{a} \cdot \mathbf{d} & \mathbf{b} \cdot \mathbf{d} \end{vmatrix}$

36 $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \times \mathbf{b} \cdot \mathbf{d})\mathbf{c} - (\mathbf{a} \times \mathbf{b} \cdot \mathbf{c})\mathbf{d}$

37 $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a}) = (\mathbf{a} \cdot \mathbf{b} \times \mathbf{c})^2$

EXERCISES 10.5

Exer. 1–4: Find parametric equations for the line through P parallel to a.

1 $P(4, 2, -3)$; $\mathbf{a} = \langle \frac{1}{3}, 2, \frac{1}{2} \rangle$

2 $P(5, 0, -2)$; $\mathbf{a} = \langle -1, -4, 1 \rangle$

3 $P(0, 0, 0)$; $\mathbf{a} = \mathbf{j}$

4 $P(1, 2, 3)$; $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

Exer. 5–8: Find parametric equations for the line through P_1 and P_2. Determine (if possible) the points at which the line intersects each of the coordinate planes.

5 $P_1(5, -2, 4)$, $P_2(2, 6, 1)$

6 $P_1(-3, 1, -1)$, $P_2(7, 11, -8)$

7 $P_1(2, 0, 5)$, $P_2(-6, 0, 3)$

8 $P_1(2, -2, 4)$, $P_2(2, -2, -3)$

9 If l has parametric equations $x = 5 - 3t$, $y = -2 + t$, $z = 1 + 9t$, find parametric equations for the line through $P(-6, 4, -3)$ that is parallel to l.

10 Find parametric equations for the line through the point $P(4, -1, 0)$ that is parallel to the line through the points $P_1(-3, 9, -2)$ and $P_2(5, 7, -3)$.

Exer. 11–14: Determine whether the two lines intersect, and if so, find the point of intersection.

11 $x = 1 + 2t$, $y = 1 - 4t$, $z = 5 - t$
$x = 4 - v$, $y = -1 + 6v$, $z = 4 + v$

12 $x = 1 - 6t$, $y = 3 + 2t$, $z = 1 - 2t$
$x = 2 + 2v$, $y = 6 + v$, $z = 2 + v$

13 $x = 3 + t$, $y = 2 - 4t$, $z = t$
$x = 4 - v$, $y = 3 + v$, $z = -2 + 3v$

14 $x = 2 - 5t$, $y = 6 + 2t$, $z = -3 - 2t$
$x = 4 - 3v$, $y = 7 + 5v$, $z = 1 + 4v$

Exer. 15–18: Equations for two lines l_1 and l_2 are given. Find the angles between l_1 and l_2.

15 $x = 7 - 2t$, $y = 4 + 3t$, $z = 5t$
$x = -1 + 4t$, $y = 3 + 4t$, $z = 1 + t$

16 $x = 5 + 3t$, $y = 4 - t$, $z = 3 + 2t$
$x = -t$, $y = 1 - 2t$, $z = 3 + t$

17 $\dfrac{x-1}{-3} = \dfrac{y+2}{8} = \dfrac{z}{-3}$; $\dfrac{x+2}{10} = \dfrac{y}{10} = \dfrac{z-4}{-7}$

18 $\dfrac{x}{3} = \dfrac{y-2}{3} = z - 1$; $\dfrac{x+5}{4} = \dfrac{y-1}{-3} = \dfrac{z+7}{-9}$

Exer. 19–26: Find an equation of the plane that satisfies the stated conditions.

19 Through $P(6, -7, 4)$ and parallel to
(a) the xy-plane **(b)** the yz-plane **(c)** the xz-plane

20 Through $P(-2, 5, -8)$ with normal vector
(a) $\mathbf{i}$ **(b)** $\mathbf{j}$ **(c)** $\mathbf{k}$

21 Through $P(-11, 4, -2)$ with normal vector $\mathbf{a} = 6\mathbf{i} - 5\mathbf{j} - \mathbf{k}$

22 Through $P(4, 2, -9)$ with normal vector $\overrightarrow{OP}$

23 Through $P(2, 5, -6)$ and parallel to the plane $3x - y + 2z = 10$

24 Through the origin and parallel to the plane $x - 6y + 4z = 7$

25 Through $P(-4, 1, 6)$ and having the same trace in the xz-plane as the plane $x + 4y - 5z = 8$

26 Through the origin and the points $P(0, 2, 5)$ and $Q(1, 4, 0)$

Exer. 27–28: Find an equation of the plane through P, Q, and R.

27 $P(1, 1, 3)$, $Q(-1, 3, 2)$, $R(1, -1, 2)$

28 $P(3, 2, 1)$, $Q(-1, 1, -2)$, $R(3, -4, 1)$

Exer. 29–36: Sketch the graph of the equation in an xyz-coordinate system.

29 **(a)** $x = 3$ **(b)** $y = -2$ **(c)** $z = 5$

30 **(a)** $x = -4$ **(b)** $y = 0$ **(c)** $z = -\frac{2}{3}$

31 $2x + y - 6 = 0$

32 $3x - 2z - 24 = 0$

33 $2y - 3z - 9 = 0$

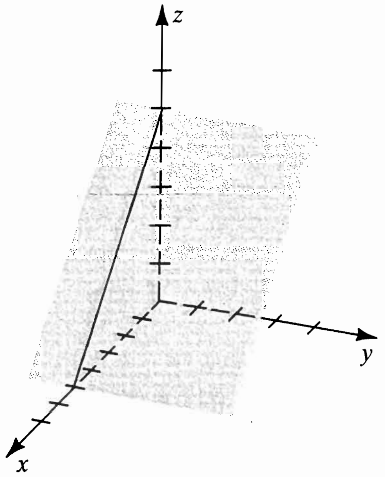
34 $5x + y - 4z + 20 = 0$

35 $2x - y + 5z + 10 = 0$

36 $x + y + z = 0$

Exer. 37–40: Find an equation of the plane.

37



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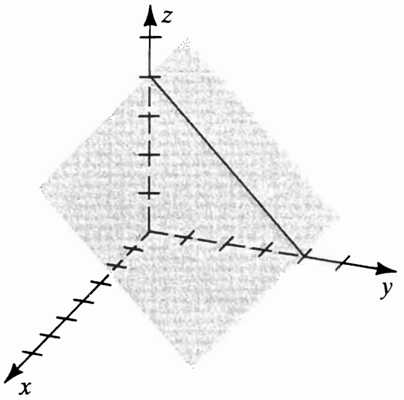

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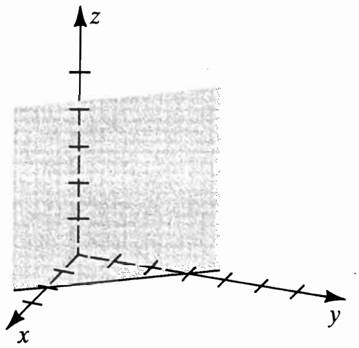

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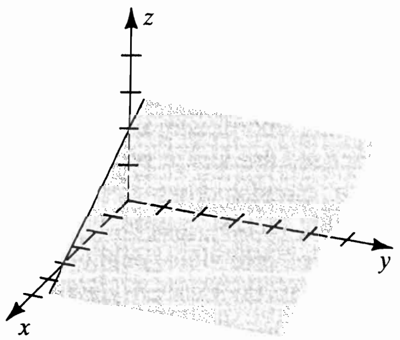


Exer. 41–42: Find an equation of the plane through P that is parallel to the given plane.

41 $P(1, 2, -3)$; $4x - y + 3z - 7 = 0$

42 $P(3, -2, 4)$; $-2x + 3y - z + 5 = 0$

Exer. 43–46: Find a symmetric form for the line through P_1 and P_2.

43 $P_1(5, -2, 4)$, $P_2(2, 6, 1)$

44 $P_1(-3, 1, -1)$, $P_2(7, 11, -8)$

45 $P_1(4, 2, -3)$, $P_2(-3, 2, 5)$

46 $P_1(5, -7, 4)$, $P_2(-2, -1, 4)$

Exer. 47–50: Find parametric equations for the line of intersection of the two planes.

47 $x + 2y - 9z = 7$, $2x - 3y + 17z = 0$

48 $2x + 5y + 16z = 13$, $-x - 2y - 6z = -5$

49 $-2x + 3y + 9z = 12$, $x - 2y - 5z = -8$

50 $5x - y - 12z = 15$, $2x + 3y + 2z = 6$

Exer. 51–52: Refer to Example 13. Find the distance from P to the plane.

51 $P(1, -1, 2)$; $3x - 7y + z - 5 = 0$

52 $P(3, 1, -2)$; $2x + 4y - 5z + 1 = 0$

Exer. 53–54: Show that the two planes are parallel and find the distance between the planes.

53 $4x - 2y + 6z = 3$, $-6x + 3y - 9z = 4$

54 $3x + 12y - 6z = -2$, $5x + 20y - 10z = 7$

Exer. 55–56: Refer to Example 14. Let l_1 be the line through A and B, and let l_2 be the line through C and D. Find the shortest distance between l_1 and l_2.

55 $A(1, -2, 3)$, $B(2, 0, 5)$; $C(4, 1, -1)$, $D(-2, 3, 4)$

56 $A(1, 3, 0)$, $B(0, 4, 5)$; $C(-2, -1, 2)$, $D(5, 1, 0)$

Exer. 57–58: Find an equation of the plane that contains the point P and the line.

57 $P(5, 0, 2)$; $x = 3t + 1$, $y = -2t + 4$, $z = t - 3$

58 $P(4, -3, 0)$; $x = t + 5$, $y = 2t - 1$, $z = -t + 7$

Exer. 59–60: Use a dot product to find the distance from A to the line through B and C.

59 $A(2, -6, 1)$; $B(3, 4, -2)$, $C(7, -1, 5)$

60 $A(1, 5, 0)$; $B(-2, 1, -4)$, $C(0, -3, 2)$

Exer. 61–62: Find the distance from the point P to the line.

61 $P(2,\ 1,\ -2);\ x = 3 - 2t,\ \ y = -4 + 3t,\ \ z = 1 + 2t$

62 $P(3,\ 1,\ -1);\ x = 1 + 4t,\ \ y = 3 - t,\ \ z = 3t$

Exer. 63–64: If a plane has nonzero x-, y-, and z-intercepts a, b, and c, respectively, then its *intercept form* is

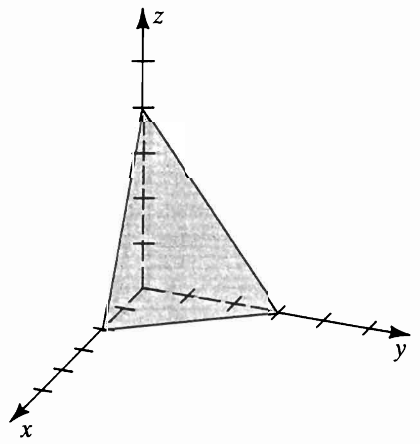
$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

Find the intercept form for the given plane.

63 $10x - 15y + 6z = 30$ **64** $12x + 15y - 20z = 60$

Exer. 65–66: Find an equation for the plane of the form $Ax + By + Cz = D$.

65



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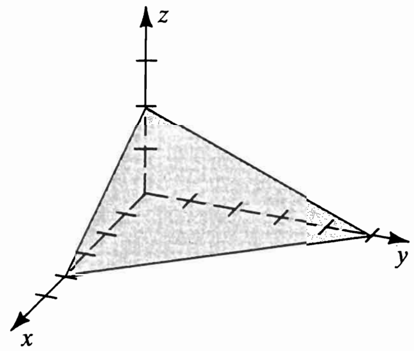


c **Exer. 67–68: Graph f and g on the same coordinate plane for $-2 \le x \le 2$. (a) Estimate the coordinates of their point P of intersection. (b) Approximate the angles between the tangent lines to the graphs at P.**

67 $f(x) = \sin(x^2),\qquad g(x) = \cos x - x$

68 $f(x) = 1 - 3x + x^3,\quad g(x) = x^5 + \frac{1}{2}$

10.6 SURFACES

In this section, we examine some techniques for obtaining accurate sketches of surfaces in space that are described by equations in x, y, and z. To sketch a surface with pencil and paper, we usually choose the coordinate axes as in Figure 10.21, regarding the y- and z-axes as lying in the plane of the paper and the x-axis as projecting out from the paper. This technique is illustrated in Figures 10.58–10.68 and the charts in this section. A disadvantage of choosing coordinate axes in this way is that when a specific equation is graphed, the shape of the resulting surface may seem distorted. For example, circular cross-sections may appear to be elliptical and vice versa. For this reason, graphs in three dimensions that are illustrated later in the text are computer-generated, with axes and units of distance chosen to provide an undistorted view of a surface. In

By Theorem (5.19), the surface area is given by

$$S = \int_0^h 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx.$$

Now $f'(x) = 2\sqrt{p}(\frac{1}{2})x^{-1/2} = \sqrt{p}/\sqrt{x}$, and so

$$\sqrt{1 + [f'(x)]^2} = \sqrt{1 + \frac{p}{x}} = \frac{\sqrt{x+p}}{\sqrt{x}},$$

which gives

$$f(x)\sqrt{1 + [f'(x)]^2} = 2\sqrt{p}\sqrt{x}\frac{\sqrt{x+p}}{\sqrt{x}} = 2\sqrt{p}\sqrt{x+p}.$$

Thus the surface area is

$$S = 2\pi 2\sqrt{p}\int_0^h \sqrt{x+p}\,dx = 4\pi\sqrt{p}\left[\frac{2}{3}(x+p)^{3/2}\right]_0^h$$

$$= \frac{8\pi\sqrt{p}}{3}\left[(h+p)^{3/2} - p^{3/2}\right] = \frac{8\pi\sqrt{p}}{3}\left\{p^{3/2}\left[\left(\frac{h+p}{p}\right)^{3/2} - 1\right]\right\}$$

$$= \frac{8\pi p^2}{3}\left[\left(1 + \frac{h}{p}\right)^{3/2} - 1\right].$$

Since the point $P(h, a)$ lies on the parabola $y^2 = 4px$, we have $a^2 = 4ph$, and we can express the altitude h in terms of the focal length p and the radius of the base a:

$$h = \frac{a^2}{4p}$$

Thus, $h/p = a^2/4p^2$, and the surface area may be written as

$$S = \frac{8\pi p^2}{3}\left[\left(1 + \frac{a^2}{4p^2}\right)^{3/2} - 1\right].$$

(b) Using the final formula of part (a) with $a = \frac{305}{2}$ and $p = 100$, we have

$$\text{the surface area } = \frac{8\pi(100^2)}{3}\left[\left(1 + \frac{152.5^2}{4(100^2)}\right)^{3/2} - 1\right] \approx 82{,}828\ \text{m}^2.$$

EXERCISES 10.6

Exer. 1–8: Sketch the graph of the cylinder in an *xyz*-coordinate system.

1 $x^2 + y^2 = 9$

2 $y^2 + z^2 = 16$

3 $4y^2 + 9z^2 = 36$

4 $x^2 + 5z^2 = 25$

5 $x^2 = 9z$

6 $x^2 - 4y = 0$

7 $y^2 - x^2 = 16$

8 $xz = 1$

Exer. 9 – 20: Match each graph with one of the equations.

A. $\frac{x^2}{25}+\frac{y^2}{9}+\frac{z^2}{4}=1$ **B.** $x=z^2+\frac{y^2}{4}$ **C.** $y^2+z^2-x^2=1$ **D.** $\frac{x^2}{4}+\frac{y^2}{9}-\frac{z^2}{4}=0$

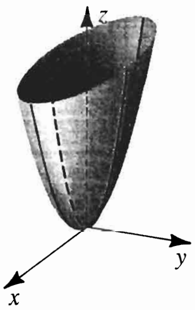
E. $z=\frac{x^2}{9}-\frac{y^2}{4}$ **F.** $z^2-\frac{x^2}{4}-y^2=1$ **G.** $\frac{z^2}{9}+\frac{y^2}{4}-\frac{x^2}{4}=0$ **H.** $\frac{x^2}{4}-y^2-z^2=1$

I. $y=\frac{x^2}{4}-\frac{z^2}{9}$ **J.** $x^2+\frac{y^2}{4}+\frac{z^2}{16}=1$ **K.** $z=\frac{x^2}{9}+y^2$ **L.** $\frac{x^2}{4}+\frac{y^2}{16}+\frac{z^2}{9}=1$

M. $y=\frac{z^2}{9}-\frac{x^2}{4}$ **N.** $y=\frac{x^2}{4}+\frac{z^2}{4}$ **O.** $z^2+\frac{x^2}{4}-y^2=1$ **P.** $\frac{x^2}{4}+\frac{z^2}{9}-\frac{y^2}{4}=0$

Q. $y^2-\frac{x^2}{4}-z^2=1$ **R.** $x^2+\frac{y^2}{4}-z^2=1$

9



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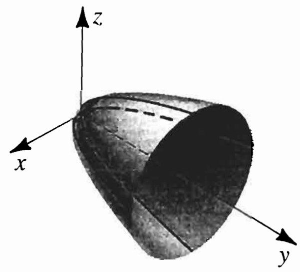

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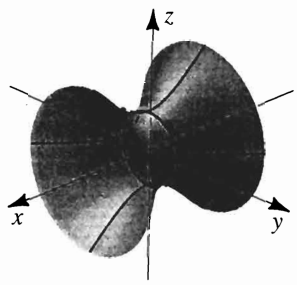

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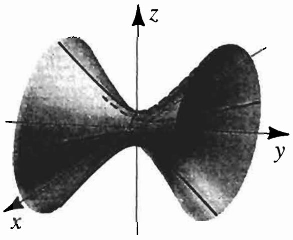

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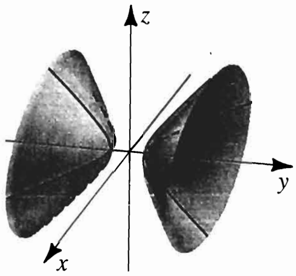

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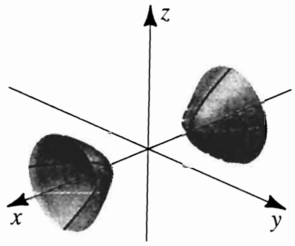

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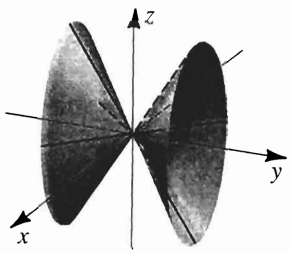

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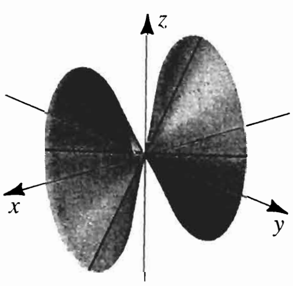

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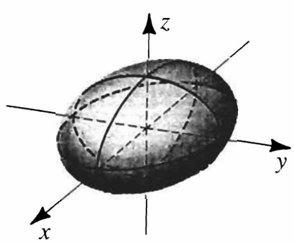

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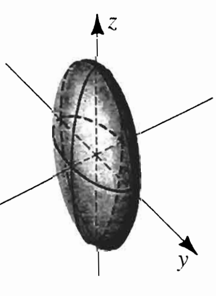

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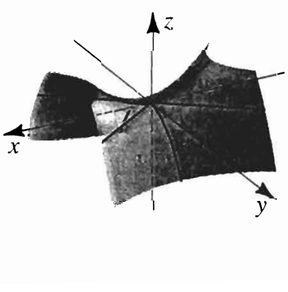

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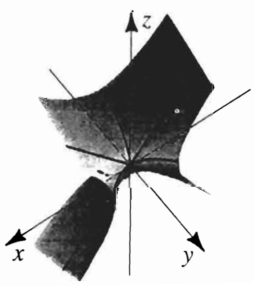


Exer. 21 – 32: Sketch the graph of the quadric surface.

Ellipsoids

21 $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$ **22** $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$

Hyperboloids of one sheet

23 **(a)** $\frac{x^2}{4} + y^2 - z^2 = 1$ **(b)** $x^2 + \frac{z^2}{4} - y^2 = 1$

24 **(a)** $z^2 + x^2 - y^2 = 1$ **(b)** $y^2 + \frac{z^2}{4} - x^2 = 1$

Hyperboloids of two sheets

25 **(a)** $x^2 - \frac{y^2}{4} - z^2 = 1$ **(b)** $\frac{z^2}{4} - y^2 - x^2 = 1$

26 **(a)** $z^2 - \frac{x^2}{4} - \frac{y^2}{4} = 1$ **(b)** $\frac{y^2}{4} - x^2 - \frac{z^2}{9} = 1$

Cones

27 **(a)** $\frac{x^2}{9} + \frac{y^2}{4} = \frac{z^2}{4}$ **(b)** $\frac{x^2}{4} - y^2 + \frac{z^2}{9} = 0$

28 **(a)** $\frac{x^2}{25} + \frac{y^2}{9} - z^2 = 0$ **(b)** $x^2 = 4y^2 + z^2$

Paraboloids

29 **(a)** $y = \frac{x^2}{4} + \frac{z^2}{9}$ **(b)** $x^2 + \frac{y^2}{4} - z = 0$

30 **(a)** $z = x^2 + \frac{y^2}{9}$ **(b)** $\frac{z^2}{25} + \frac{y^2}{9} - x = 0$

Hyperbolic paraboloids

31 **(a)** $z = x^2 - y^2$ **(b)** $z = y^2 - x^2$

32 **(a)** $z = \frac{y^2}{9} - \frac{x^2}{4}$ **(b)** $z = \frac{x^2}{4} - \frac{y^2}{9}$

Exer. 33 – 46: Sketch the graph of the equation in an *xyz*-coordinate system, and identify the surface.

33 $16x^2 - 4y^2 - z^2 + 1 = 0$

34 $8x^2 + 4y^2 + z^2 = 16$ **35** $36x = 9y^2 + z^2$

36 $16x^2 + 100y^2 - 25z^2 = 400$

37 $x^2 - 16y^2 = 4z^2$ **38** $3x^2 - 4y^2 - z^2 = 12$

39 $9x^2 + 4y^2 + z^2 = 36$ **40** $16y = x^2 + 4z^2$

41 $z = e^y$ **42** $x^2 + (y-2)^2 = 1$

43 $4x - 3y = 12$ **44** $2x + 4y + 3z = 12$

45 $y^2 - 9x^2 - z^2 - 9 = 0$ **46** $36x^2 - 16y^2 + 9z^2 = 0$

c **Exer. 47 – 50: Graph the surface.**

47 $z = \frac{1}{5}y^2 - 3|x|$ **48** $z = x^2 + 3xy + 4y^2$

49 $z = xy + x^2$ **50** $z = \frac{y^2}{16} - \frac{xy}{15} - \frac{x^2}{9}$

Exer. 51 – 56: Find an equation of the surface obtained by revolving the graph of the equation about the indicated axis.

51 $x^2 + 4y^2 = 16$; y-axis **52** $y^2 = 4x$; x-axis

53 $z = 4 - y^2$; z-axis **54** $z = e^{-y^2}$; y-axis

55 $z^2 - x^2 = 1$; x-axis **56** $xz = 1$; z-axis

57 Although we often use a sphere as a model of the earth, a more precise relationship is needed for surveying the earth's surface. The *Clarke ellipsoid* (1866), with equation $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$ for $a = b = 6378.2064$ km and $c = 6356.5838$ km, is used to establish the geographic positions of control points in the U.S. national geodetic network.

(a) Explain briefly the difference between the Clarke ellipsoid and the usual spherical representation of the earth's surface.

(b) Curves of equal latitude are traces in a plane of the form $z = k$. Describe these curves.

(c) Curves of equal longitude (or *meridians*) are traces in a plane of the form $y = mx$. Describe these curves.

58 Let m and b be nonzero real numbers.

(a) If the line $y = mx + b$ intersects the parabola $y^2 = 4px$ in only one point, show that $p = mb$.

(b) Show that the slope of the tangent line to $y^2 = 4px$ at $P(x_1, y_1)$ is $y_1/(2x_1)$.

59 Establish the reflective property of the parabola. (*Hint:* Show that $d(Q, F) = d(F, P)$ in Figure 10.76 and use the result in Exercise 58.)

EXERCISES 11.1

Exer. 1–8: (a) Sketch the two vectors listed after the formula for r(t). (b) Sketch, on the same coordinate system, the curve C determined by r(t), and indicate the orientation for the given values of t.

1 $\mathbf{r}(t) = 3t\mathbf{i} + (1 - 9t^2)\mathbf{j}$, $\mathbf{r}(0)$, $\mathbf{r}(1)$; t in $\mathbb{R}$

2 $\mathbf{r}(t) = (1 - t^3)\mathbf{i} + t\mathbf{j}$, $\mathbf{r}(1)$, $\mathbf{r}(2)$; $t \geq 0$

3 $\mathbf{r}(t) = (t^3 - 1)\mathbf{i} + (t^2 + 2)\mathbf{j}$, $\mathbf{r}(1)$, $\mathbf{r}(2)$; $-2 \leq t \leq 2$

4 $\mathbf{r}(t) = (2 + \cos t)\mathbf{i} - (3 - \sin t)\mathbf{j}$, $\mathbf{r}(\pi/2)$, $\mathbf{r}(\pi)$; $0 \leq t \leq 2\pi$

5 $\mathbf{r}(t) = (3 + t)\mathbf{i} + (2 - t)\mathbf{j} + (1 + 2t)\mathbf{k}$, $\mathbf{r}(-1)$, $\mathbf{r}(0)$; $t \geq -1$

6 $\mathbf{r}(t) = t\mathbf{i} - 3\sin t\mathbf{j} + 3\cos t\mathbf{k}$, $\mathbf{r}(0)$, $\mathbf{r}(\pi/2)$; $t \geq 0$

7 $\mathbf{r}(t) = t\mathbf{i} + 4\cos t\mathbf{j} + 9\sin t\mathbf{k}$, $\mathbf{r}(0)$, $\mathbf{r}(\pi/2)$; $t \geq 0$

8 $\mathbf{r}(t) = \tan t\mathbf{i} + \sec t\mathbf{j} + 2\mathbf{k}$, $\mathbf{r}(0)$, $\mathbf{r}(\pi/4)$; $-\pi/2 < t < \pi/2$

Exer. 9–16: Sketch the curve C determined by r(t), and indicate the orientation.

9 $\mathbf{r}(t) = e^t\cos t\mathbf{i} + e^t\sin t\mathbf{j}$; $0 \leq t \leq \pi$

10 $\mathbf{r}(t) = 2\cosh t\mathbf{i} + 3\sinh t\mathbf{j}$; t in $\mathbb{R}$

11 $\mathbf{r}(t) = t\mathbf{i} + 2t^2\mathbf{j} + 3t^3\mathbf{k}$; t in $\mathbb{R}$

12 $\mathbf{r}(t) = t^3\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$; $0 \leq t \leq 4$

13 $\mathbf{r}(t) = (t^2 + 1)\mathbf{i} + t\mathbf{j} + 3\mathbf{k}$; t in $\mathbb{R}$

14 $\mathbf{r}(t) = 6\sin t\mathbf{i} + 4\mathbf{j} + 25\cos t\mathbf{k}$; $-2\pi \leq t \leq 2\pi$

15 $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + \sin t\mathbf{k}$; t in $\mathbb{R}$

16 $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + e^t\mathbf{k}$; t in $\mathbb{R}$

c **Exer. 17–20: Plot the curve C determined by r(t), and indicate the orientation.**

17 $\mathbf{r}(t) = 3\sin(t^2)\mathbf{i} + (4 - t^{3/2})\mathbf{j}$; $0 \leq t \leq 5$

18 $\mathbf{r}(t) = e^{\sin 3t}\mathbf{i} + e^{-\cos t}\mathbf{j}$; $0 \leq t \leq 2\pi$

19 $\mathbf{r}(t) = (4 + \sin 2t)\mathbf{i} + (1 - 3\cos 3t)\mathbf{j}$; $-\pi \leq t \leq \pi$

20 $\mathbf{r}(t) = (1 + 3\sin 2t)\mathbf{i} + (2\cos 3t)\mathbf{j}$; $0 \leq t \leq \pi$

Exer. 21–26: Find the arc length of the parametrized curve. Estimate with numerical integration if needed, and express answers to four decimal places of accuracy.

21 $x = 5t$, $y = 4t^2$, $z = 3t^2$; $0 \leq t \leq 2$

22 $x = t^2$, $y = t\sin t$, $z = t\cos t$; $0 \leq t \leq 1$

23 $x = e^t\cos t$, $y = e^t$, $z = e^t\sin t$; $0 \leq t \leq 2\pi$

24 $x = 2t$, $y = 4\sin 3t$, $z = 4\cos 3t$; $0 \leq t \leq 2\pi$

c 25 $x = 3\cos t$, $y = 2\sin t$, $z = 2 - t^2$; $-\sqrt{2} \leq t \leq \sqrt{2}$

c 26 $x = (1 - t)^2$, $y = t^2$, $z = 2(1 - t)t$; $0 \leq t \leq 1$

27 A *concho-spiral* is a curve C that has a parametrization $x = ae^{\mu t}\cos t$, $y = ae^{\mu t}\sin t$, $z = be^{\mu t}$; $t \geq 0$, where a, b, and μ are constants.

(a) Show that C lies on the cone $a^2z^2 = b^2(x^2 + y^2)$.

(b) Sketch C for $a = b = 4$ and $\mu = -1$.

(c) Find the length of C corresponding to the t-interval $[0, \infty)$.

28 A curve C has the parametrization

$$x = a\sin t\sin\alpha,\quad y = b\sin t\cos\alpha,\quad z = c\cos t;\quad t \geq 0,$$

where a, b, c, and α are positive constants.

(a) Show that C lies on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

(b) Show that C also lies on a plane that contains the z-axis.

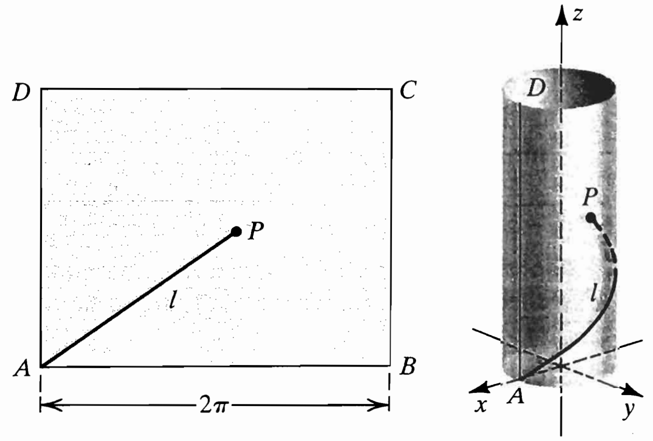
(c) Describe the curve C.

29 **(a)** Show that a twisted cubic having parametrization $x = at$, $y = bt^2$, $z = ct^3$; $t \geq 0$ intersects a given plane in at most three points.

(b) Determine the length of the twisted cubic $x = 6t$, $y = 3t^2$, $z = t^3$ between the points corresponding to $t = 0$ and $t = 1$.

30 A rectangle can be made into a cylinder by joining together two opposite and parallel edges. Shown in the figure on the following page is a rectangle $ABCD$ of width 2π. Edge AD is joined to edge BC, and point A is then positioned at $(1, 0, 0)$ to form part of the cylinder $x^2 + y^2 = 1$, sketched on the right of the rectangle.

Exercise 30



(a) If l is the line segment from A to another point P in the rectangle and m is the slope of l, show that, as a curve on the cylinder, l has a parametrization $x = \cos t,\ y = \sin t,\ z = mt$.

(b) Use part (a) to show that the curve on the cylinder with shortest length from (1, 0, 0) to another point P is a helix (see Example 4).

11.2 LIMITS, DERIVATIVES, AND INTEGRALS

Limits, derivatives, and integrals for vector-valued functions are natural generalizations of the corresponding concepts for real-valued functions of a single real variable. We consider these generalizations in this section. We could define the limit of a vector-valued function $\mathbf{r}$ by using an ϵ–δ approach similar to that used in Definition (1.4) (see Exercise 45); however, since $\mathbf{r}(t)$ may be expressed in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, where the components are scalar functions f, g, and h, it is simpler to use the following definition.

Definition 11.4

Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$. The **limit of** $\mathbf{r}(t)$ as t approaches a is

$$\lim_{t\to a} \mathbf{r}(t) = \left[\lim_{t\to a} f(t)\right]\mathbf{i} + \left[\lim_{t\to a} g(t)\right]\mathbf{j} + \left[\lim_{t\to a} h(t)\right]\mathbf{k},$$

provided f, g, and h have limits as t approaches a.

Thus, to find $\lim_{t\to a} \mathbf{r}(t)$, we take the limit of each component of $\mathbf{r}(t)$. We may state a definition similar to (11.4) for one-sided limits.

ILLUSTRATION

$$\lim_{t\to 2}(t^2\mathbf{i} + 3t\mathbf{j} + 5\mathbf{k}) = \left[\lim_{t\to 2} t^2\right]\mathbf{i} + \left[\lim_{t\to 2} 3t\right]\mathbf{j} + \left[\lim_{t\to 2} 5\right]\mathbf{k}$$
$$= 4\mathbf{i} + 6\mathbf{j} + 5\mathbf{k}$$

If $\mathbf{R}'(t) = \mathbf{r}(t)$, then $\mathbf{R}(t)$ is an **antiderivative** of $\mathbf{r}(t)$. The next result is analogous to the fundamental theorem of calculus.

Theorem 11.11

If $\mathbf{R}(t)$ is an antiderivative of $\mathbf{r}(t)$ on $[a, b]$, then

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(t)\Big]_a^b = \mathbf{R}(b) - \mathbf{R}(a).$$

EXAMPLE ■ 4 Find $\int_0^2 \mathbf{r}(t)\,dt$ if $\mathbf{r}(t) = 12t^3\mathbf{i} + 4e^{2t}\mathbf{j} + (t+1)^{-1}\mathbf{k}$.

SOLUTION Finding an antiderivative for each component of $\mathbf{r}(t)$, we obtain

$$\mathbf{R}(t) = 3t^4\mathbf{i} + 2e^{2t}\mathbf{j} + \ln(t+1)\mathbf{k}.$$

Since $\mathbf{R}'(t) = \mathbf{r}(t)$, it follows from Theorem (11.11) that

$$\begin{aligned}\int_0^2 \mathbf{r}(t)\,dt &= \mathbf{R}(2) - \mathbf{R}(0)\\ &= (48\mathbf{i} + 2e^4\mathbf{j} + \ln 3\mathbf{k}) - (0\mathbf{i} + 2\mathbf{j} + 0\mathbf{k})\\ &= 48\mathbf{i} + 2(e^4 - 1)\mathbf{j} + \ln 3\mathbf{k}.\end{aligned}$$

The theory of indefinite integrals of vector-valued functions is similar to that developed for real-valued functions in Chapter 4. The proofs of theorems require only minor modifications of those given earlier and are thus omitted. If $\mathbf{R}(t)$ is an antiderivative of $\mathbf{r}(t)$, then every antiderivative has the form $\mathbf{R}(t) + \mathbf{c}$ for some (constant) vector $\mathbf{c}$, and we write

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{c}, \quad \text{where} \quad \mathbf{R}'(t) = \mathbf{r}(t).$$

EXERCISES 11.2

Exer. 1–8: (a) Find the domain of r. (b) Find $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$.

1 $\mathbf{r}(t) = \sqrt{t-1}\,\mathbf{i} + \sqrt{2-t}\,\mathbf{j}$

2 $\mathbf{r}(t) = \dfrac{1}{t}\mathbf{i} + \sin 3t\mathbf{j}$

3 $\mathbf{r}(t) = \tan t\mathbf{i} + (t^2 + 8t)\mathbf{j}$

4 $\mathbf{r}(t) = e^{(t^2)}\mathbf{i} + \sin^{-1} t\mathbf{j}$

5 $\mathbf{r}(t) = t^2\mathbf{i} + \tan t\mathbf{j} + 3\mathbf{k}$

6 $\mathbf{r}(t) = \sqrt[3]{t}\,\mathbf{i} + \dfrac{1}{t}\mathbf{j} + e^{-t}\mathbf{k}$

7 $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + e^{2t}\mathbf{j} + t\mathbf{k}$

8 $\mathbf{r}(t) = \ln(1-t)\mathbf{i} + \sin t\mathbf{j} + t^2\mathbf{k}$

Exer. 9–16: (a) Sketch the curve (in the *xy*-plane) determined by $\mathbf{r}(t)$ and indicate the orientation. (b) Find $\mathbf{r}'(t)$ and sketch $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ for the indicated value of t.

9 $\mathbf{r}(t) = -\frac{1}{4}t^4\mathbf{i} + t^2\mathbf{j}$; $\quad t = 2$

10 $\mathbf{r}(t) = e^{2t}\mathbf{i} + e^{-4t}\mathbf{j}$; $\quad t = 0$

11 $\mathbf{r}(t) = 4\cos t\mathbf{i} + 2\sin t\mathbf{j}$; $\quad t = 3\pi/4$

12 $\mathbf{r}(t) = 2\sec t\mathbf{i} + 3\tan t\mathbf{j}$; $\quad t = \pi/4, \quad |t| < \pi/2$

13 $\mathbf{r}(t) = t^3\mathbf{i} + t^{-3}\mathbf{j}$; $\quad t = 1, \quad t > 0$

14 $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$; $\quad t = -1$

15 $\mathbf{r}(t) = (2t - 1)\mathbf{i} + (4 - t)\mathbf{j}$; $\quad t = 3$

16 $\mathbf{r}(t) = 5\mathbf{i} + t^3\mathbf{j}$; $\quad t = 2$

Exer. 17–20: A curve C is given parametrically. Find parametric equations for the tangent line to C at P.

17 $x = 2t^3 - 1,\ y = -5t^2 + 3,\ z = 8t + 2$; $\ P(1, -2, 10)$

18 $x = 4\sqrt{t}, \quad y = t^2 - 10, \quad z = 4/t$; $\quad P(8, 6, 1)$

19 $x = e^t, \quad y = te^t, \quad z = t^2 + 4$; $\ P(1, 0, 4)$

20 $x = t\sin t, \quad y = t\cos t, \quad z = t$; $\quad P(\pi/2, 0, \pi/2)$

Exer. 21–22: A curve C is given parametrically. Find two unit tangent vectors to C at P.

21 $x = e^{2t}, \quad y = e^{-t}, \quad z = t^2 + 4$; $\ P(1, 1, 4)$

22 $x = \sin t + 2, \quad y = \cos t, \quad z = t$; $\quad P(2, 1, 0)$

23 Refer to Exercise 27 of Section 11.1. Show that the concho-spiral has the special property that the angle between $\mathbf{k}$ and the tangent vector $\mathbf{r}'(t)$ is a constant.

24 The *general helix* is a curve whose tangent vector makes a constant angle with a fixed unit vector $\mathbf{u}$. Show that the curve with parametrization $x = 3t - t^3$, $y = 3t^2$, $z = 3t + t^3$; t in $\mathbb{R}$ is a general helix by finding an appropriate vector $\mathbf{u}$.

25 A point P moves along a curve C in such a way that the position vector $\mathbf{r}(t)$ of P is equal to the tangent vector $\mathbf{r}'(t)$ for every t. Find parametric equations for C, and describe the graph.

26 A point moves along a curve C in such a way that the position vector $\mathbf{r}(t)$ and the tangent vector $\mathbf{r}'(t)$ are always orthogonal. Prove that C lies on a sphere with center at the origin. (*Hint:* Show that $(d/dt)\,\|\mathbf{r}(t)\|^2 = 0$.)

Exer. 27–30: Evaluate the integral.

27 $\displaystyle\int_0^2 (6t^2\mathbf{i} - 4t\mathbf{j} + 3\mathbf{k})\,dt$

28 $\displaystyle\int_{-1}^1 (-5t\mathbf{i} + 8t^3\mathbf{j} - 3t^2\mathbf{k})\,dt$

29 $\displaystyle\int_0^{\pi/4} (\sin t\mathbf{i} - \cos t\mathbf{j} + \tan t\mathbf{k})\,dt$

30 $\displaystyle\int_0^1 [te^{(t^2)}\mathbf{i} + \sqrt{t}\mathbf{j} + (t^2 + 1)^{-1}\mathbf{k}]\,dt$

Exer. 31–34: Find $\mathbf{r}(t)$ subject to the given conditions.

31 $\mathbf{r}'(t) = t^2\mathbf{i} + (6t + 1)\mathbf{j} + 8t^3\mathbf{k}$,
$\quad \mathbf{r}(0) = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

32 $\mathbf{r}'(t) = 2\mathbf{i} - 4t^3\mathbf{j} + 6\sqrt{t}\mathbf{k}$,
$\quad \mathbf{r}(0) = \mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$

33 $\mathbf{r}''(t) = 6t\mathbf{i} - 12t^2\mathbf{j} + \mathbf{k}$,
$\quad \mathbf{r}'(0) = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}, \quad \mathbf{r}(0) = 7\mathbf{i} + \mathbf{k}$

34 $\mathbf{r}''(t) = 6t\mathbf{i} + 3\mathbf{j}$,
$\quad \mathbf{r}'(0) = 4\mathbf{i} - \mathbf{j} + \mathbf{k}, \quad \mathbf{r}(0) = 5\mathbf{j}$

Exer. 35–36: If a curve C has a tangent vector a at a point P, then the *normal plane* to C at P is the plane through P with normal vector a. Find an equation of the normal plane to the given curve at P.

35 $x = e^t, \quad y = te^t, \quad z = t^2 + 4$; $\ P(1, 0, 4)$

36 $x = t\sin t, \quad y = t\cos t, \quad z = t$; $\quad P(\pi/2, 0, \pi/2)$

Exer. 37–38: Find $[\mathbf{u}(t)\cdot\mathbf{v}(t)]'$ and $[\mathbf{u}(t)\times\mathbf{v}(t)]'$.

37 $\mathbf{u}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}, \quad \mathbf{v}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + 2\sin t\mathbf{k}$

38 $\mathbf{u}(t) = 2t\mathbf{i} + 6t\mathbf{j} + t^2\mathbf{k}, \quad \mathbf{v}(t) = e^{-t}\mathbf{i} - e^{-t}\mathbf{j} + \mathbf{k}$

39 If $\mathbf{u}$ and $\mathbf{v}$ are vector-valued functions that have limits as $t \to a$, prove the following:

(a) $\lim\limits_{t\to a}[\mathbf{u}(t) + \mathbf{v}(t)] = \lim\limits_{t\to a}\mathbf{u}(t) + \lim\limits_{t\to a}\mathbf{v}(t)$

(b) $\lim\limits_{t\to a}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \lim\limits_{t\to a}\mathbf{u}(t)\cdot\lim\limits_{t\to a}\mathbf{v}(t)$

(c) $\lim\limits_{t\to a} c\mathbf{u}(t) = c\lim\limits_{t\to a}\mathbf{u}(t)$, where c is a scalar

40 If a scalar function f and a vector-valued function $\mathbf{u}$ have limits as $t \to a$, prove that

$$\lim_{t\to a} f(t)\mathbf{u}(t) = \left[\lim_{t\to a} f(t)\right]\left[\lim_{t\to a}\mathbf{u}(t)\right].$$

41 Prove that $\lim_{t\to a}\mathbf{u}(t) = \mathbf{b}$ if and only if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\|\mathbf{u}(t) - \mathbf{b}\| < \epsilon$ whenever $0 < |t - a| < \delta$. Give a graphical description of this result.

42 If $\mathbf{u}$ and $\mathbf{v}$ have limits as $t \to a$, prove that

$$\lim_{t\to a}[\mathbf{u}(t)\times\mathbf{v}(t)] = \left[\lim_{t\to a}\mathbf{u}(t)\right]\times\left[\lim_{t\to a}\mathbf{v}(t)\right].$$

Exer. 43–44: If u and v are differentiable, prove the stated rule for derivatives.

43 $[\mathbf{u}(t) + \mathbf{v}(t)]' = \mathbf{u}'(t) + \mathbf{v}'(t)$

44 $[\mathbf{u}(t)\times\mathbf{v}(t)]' = \mathbf{u}(t)\times\mathbf{v}'(t) + \mathbf{u}'(t)\times\mathbf{v}(t)$

45 If f and $\mathbf{u}$ are differentiable, prove that

$$[f(t)\mathbf{u}(t)]' = f(t)\mathbf{u}'(t) + f'(t)\mathbf{u}(t).$$

46 If f and $\mathbf{u}$ are differentiable with suitably restricted domains, prove the chain rule

$$[\mathbf{u}(f(t))]' = f'(t)\mathbf{u}'(f(t)).$$

47 If **u**, **v**, and **w** are differentiable, prove that

$$[\mathbf{u}(t) \cdot \mathbf{v}(t) \times \mathbf{w}(t)]' = [\mathbf{u}'(t) \cdot \mathbf{v}(t) \times \mathbf{w}(t)] + [\mathbf{u}(t) \cdot \mathbf{v}'(t) \times \mathbf{w}(t)] + [\mathbf{u}(t) \cdot \mathbf{v}(t) \times \mathbf{w}'(t)].$$

48 If $\mathbf{u}'(t)$ and $\mathbf{u}''(t)$ exist, prove that

$$[\mathbf{u}(t) \times \mathbf{u}'(t)]' = \mathbf{u}(t) \times \mathbf{u}''(t).$$

49 If **u** and **v** are integrable on $[a, b]$ and if c is a scalar, prove the following:

(a) $\displaystyle\int_a^b [\mathbf{u}(t) + \mathbf{v}(t)]\,dt = \int_a^b \mathbf{u}(t)\,dt + \int_a^b \mathbf{v}(t)\,dt$

(b) $\displaystyle\int_a^b c\mathbf{u}(t)\,dt = c\int_a^b \mathbf{u}(t)\,dt$

50 If **u** is integrable on $[a, b]$ and **c** is in V_3, prove that

$$\int_a^b \mathbf{c} \cdot \mathbf{u}(t)\,dt = \mathbf{c} \cdot \int_a^b \mathbf{u}(t)\,dt.$$

11.3 CURVILINEAR MOTION

In Section 3.7, we analyzed the motion of an object moving along a straight line using a function f whose value $f(t)$ was the position of the object at time t. In this section, we investigate the motion of an object along a curve in the plane or in space. Motion often takes place in a plane. For example, although the earth moves through space, its orbit lies in a plane. (This will be proved in Section 11.6.) To study the motion of a point P in a coordinate plane, it is essential to know its position (x, y) at every instant. As usual, for *objects* in motion we assume that the mass is concentrated at P. Suppose the coordinates of P are given by the parametric equations

$$x = f(t), \quad y = g(t),$$

where t is in some interval I. If we let

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j},$$

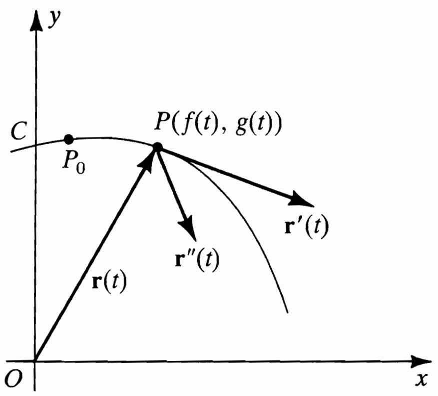
then as t varies through I, the endpoint of $\mathbf{r}(t)$ traces the path C of the point. We refer to $\mathbf{r}(t)$ as the **position vector of *P***. As in Figure 11.14, we represent

$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j}$$

as a tangent vector to C with initial point P. The vector $\mathbf{r}'(t)$ points in the direction of increasing values of t and has magnitude

$$\|\mathbf{r}'(t)\| = \sqrt{[f'(t)]^2 + [g'(t)]^2}.$$

Figure 11.14



Let t_0 be any number in I, and let P_0 be the point on C that corresponds to t_0 (see Figure 11.14). If C is smooth, then, by Theorem (9.6), the arc length $s(t)$ of C from P_0 to P is

$$s(t) = \int_{t_0}^{t} \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt = \int_{t_0}^{t} \|\mathbf{r}'(t)\|\,dt.$$

Applying Theorem (4.35), we obtain

$$\frac{d}{dt}[s(t)] = \frac{d}{dt}\int_{t_0}^{t} \|\mathbf{r}'(t)\|\,dt = \|\mathbf{r}'(t)\|.$$

for the projectile to reach D. Substituting this value of t into the parametric equation for y found in part (a), we find that the maximum altitude is

$$-\frac{1}{2}g\left(\frac{v_0 \sin\alpha}{g}\right)^2 + (v_0 \sin\alpha)\left(\frac{v_0 \sin\alpha}{g}\right) = \frac{v_0^2 \sin^2\alpha}{2g}.$$

EXERCISES 11.3

Exer. 1–8: Let r(t) be the position vector of a moving point P. Sketch the path C of P together with v(t) and a(t) for the given value of t.

1 $\mathbf{r}(t) = 2t\mathbf{i} + (4t^2 + 1)\mathbf{j}$; $t = 1$

2 $\mathbf{r}(t) = (4 - 9t^2)\mathbf{i} + 3t\mathbf{j}$; $t = 1$

3 $\mathbf{r}(t) = \sin t\mathbf{i} + 4\cos 2t\mathbf{j}$; $t = \pi/6$

4 $\mathbf{r}(t) = \cos^2 t\mathbf{i} + 2\sin t\mathbf{j}$; $t = 3\pi/4$

5 $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$; $t = \pi/2$

6 $\mathbf{r}(t) = 4\sin t\mathbf{i} + 2t\mathbf{j} + 9\cos t\mathbf{k}$; $t = 3\pi/4$

7 $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + 2t\mathbf{k}$; $t = 1$

8 $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j} + t\mathbf{k}$; $t = 2$

Exer. 9–16: If r(t) is the position vector of a moving point P, find its velocity, acceleration, and speed at the given time t.

9 $\mathbf{r}(t) = \frac{2}{t}\mathbf{i} + \frac{3}{t+1}\mathbf{j}$; $t = 2$

10 $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + (1 + \sqrt{t})\mathbf{j}$; $t = 4$

11 $\mathbf{r}(t) = e^{2t}\mathbf{i} + e^{-t}\mathbf{j}$; $t = 0$

12 $\mathbf{r}(t) = 2t\mathbf{i} + e^{-t^2}\mathbf{j}$; $t = 1$

13 $\mathbf{r}(t) = e^t(\cos t\mathbf{i} + \sin t\mathbf{j} + \mathbf{k})$; $t = \pi/2$

14 $\mathbf{r}(t) = t(\cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k})$; $t = \pi/2$

15 $\mathbf{r}(t) = (1 + t)\mathbf{i} + 2t\mathbf{j} + (2 + 3t)\mathbf{k}$; $t = 2$

16 $\mathbf{r}(t) = 2t\mathbf{i} + \mathbf{j} + 9t^2\mathbf{k}$; $t = 2$

17 If a point moves at a constant speed, prove that the velocity and acceleration vectors are orthogonal. (*Hint:* See Theorem (11.9).)

18 If the acceleration of a moving point is always **0**, prove that the motion is along a line.

c **Exer. 19–20: Let r(t) be the position vector of a moving point. Plot the path C traced out by the point (preferably in a dot or a point mode). From the graph, estimate the value of t for which the velocity is largest.**

19 $\mathbf{r}(t) = 28\cos t\mathbf{i} + 10\sin(0.8t)\mathbf{j}$; $0 \le t \le 2\pi$

20 $\mathbf{r}(t) = (5 + \cos 3t)\mathbf{i} + 1.6\sin 2t\mathbf{j}$; $0 \le t \le \pi$

Exer. 21–22: Solve, using the results of Example 2 and 4000 mi for the radius of the earth.

21 A space shuttle is in a circular orbit 150 mi above the surface of the earth. Approximate

(a) its speed

(b) the time required for one revolution

22 An earth satellite is in a circular orbit. If the time required for one revolution is 88 min, approximate the satellite's altitude.

Exer. 23–26: Solve, using the results of Example 5.

23 A projectile is fired from level ground with an initial speed of 1500 ft/sec and angle of elevation 30°. Find

(a) the velocity at time t

(b) the maximum altitude

(c) the range

(d) the speed at which the projectile strikes the ground

24 Work Exercise 23 if the angle of elevation is 60°.

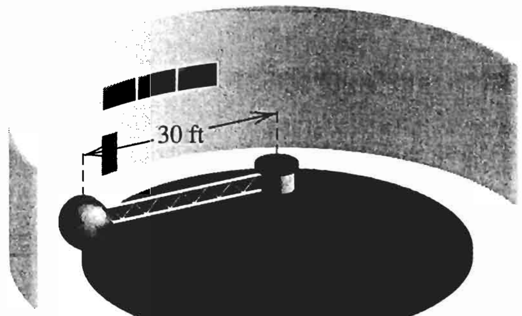
25 A baseball player throws a ball a distance of 250 ft. If the ball is released at an angle of 45° with the horizontal, find its initial speed.

26 A projectile is fired horizontally with a velocity of 1800 ft/sec from an altitude of 1000 ft above level ground. When and where does it strike the ground?

27 To test ability to withstand G-forces, an astronaut is placed at the end of a centrifuge device (see figure) that

rotates at an angular velocity ω. If the arm is 30 ft in length, find the number of revolutions per second that will result in an acceleration that is eight times that of gravity $\mathbf{g}$. (Use $\|\mathbf{g}\| = 32$ ft/sec^2.)

Exercise 27



28 The orbits of Earth, Venus, and Neptune are nearly circular. Given the information in the table, estimate the (average) speed of each planet to the nearest 0.1 km/sec.

Planet	Period (days)	Distance from sun (10^6 km)
Earth	365.3	149.6
Venus	224.7	108.2
Neptune	60,188	4498

29 A satellite moves in a circular orbit about the earth at a distance of d miles from the earth's surface. The magnitude of the force of attraction $\mathbf{F}$ between the satellite and the earth is

$$\|\mathbf{F}\| = G\frac{mM}{(R+d)^2},$$

where m is the mass of the satellite, M is the mass of the earth, R is the radius of the earth, and G is a gravitational constant. Use the results of Example 2 to establish *Kepler's third law* for circular orbits:

$$T^2 = \frac{4\pi^2}{GM}(R+d)^3,$$

where T is the period of the satellite. (*Hint:* $\|\mathbf{F}\|$ is also given by $m\,\|\mathbf{r}''(t)\|$.)

c **30** Refer to Exercise 29. If the period of a satellite is measured in days, Kepler's third law for circular orbits may be written $T^2 = 0.00346[1 + (d/R)]^3$.

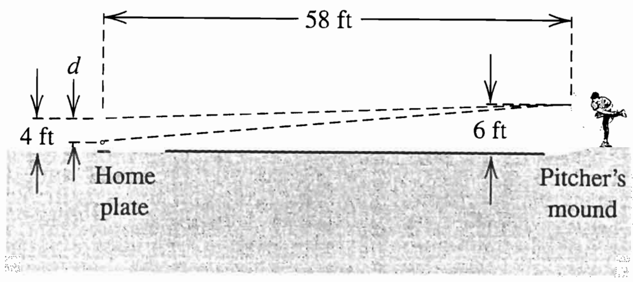
(a) A satellite is moving in a circular orbit that is 1000 mi above the earth's surface. Assuming that the radius of the earth is 3959 mi, estimate the period of the satellite's orbit to the nearest 0.01 hr.

(b) In a *geosynchronous orbit*, a satellite is always located in the same position relative to the earth—that is, the period of the satellite is one day. Estimate, to the nearest mile, the distance of such a satellite from the earth's surface. (Information of this type is needed for positioning communication satellites.)

Exer. 31–32: Solve, using the results of Example 6.

31 A major-league pitcher releases a ball at a point 6 ft above the ground and 58 ft from home plate at a speed of 100 mi/hr. If gravity had no effect, the ball would travel along a line and cross home plate 4 ft off the ground, as shown in the figure. Find the drop d caused by gravity.

Exercise 31



32 A quarterback on a football team throws a pass, releasing the ball at an angle of 30° with the horizontal. Approximate the velocity at which the football must be released to reach a receiver 150 ft downfield. (Neglect air resistance.)

33 Vertical wind shear in the lowest 300 ft of the atmosphere is of great importance to aircraft during takeoffs and landings. Vertical wind shear is defined as $d\mathbf{v}/dh$, where $\mathbf{v}$ is the wind velocity and h is the height above the ground. During strong wind gusts at a certain airport, the wind velocity (in miles per hour) for altitudes h between 0 and 200 ft is estimated to be

$$\mathbf{v} = (12 + 0.006h^{3/2})\mathbf{i} + (10 + 0.005h^{3/2})\mathbf{j}.$$

Calculate the magnitude of the vertical wind shear 150 ft above ground.

The formula for K in Definition (11.20) is usually cumbersome to apply to specific problems. In the next section, we shall derive a more practical formula that can be used to find curvature.

EXERCISES 11.4

Exer. 1–6: (a) Find the unit tangent and normal vectors T(*t*) and N(*t*) for the curve *C* determined by r(*t*). (b) Sketch the graph of *C*, and show T(*t*) and N(*t*) for the given value of *t*.

1 $\mathbf{r}(t) = t\mathbf{i} - \frac{1}{2}t^2\mathbf{j}$; $t = 1$

2 $\mathbf{r}(t) = -t^2\mathbf{i} + 2t\mathbf{j}$; $t = 1$

3 $\mathbf{r}(t) = t^3\mathbf{i} + 3t\mathbf{j}$; $t = 1$

4 $\mathbf{r}(t) = (4 + \cos t)\mathbf{i} - (3 - \sin t)\mathbf{j}$; $t = \pi/6$

5 $\mathbf{r}(t) = 2\sin t\mathbf{i} + 3\mathbf{j} + 2\cos t\mathbf{k}$; $t = \pi/4$

6 $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{2}t^2\mathbf{j} + t^2\mathbf{k}$; $t = 1$

Exer. 7–18: Find the curvature of the curve at *P*.

7 $y = 2 - x^3$; $P(1, 1)$

8 $y = x^4$; $P(1, 1)$

9 $y = e^{(x^2)}$; $P(0, 1)$

10 $y = \ln(x - 1)$; $P(2, 0)$

11 $y = \cos 2x$; $P(0, 1)$

12 $y = \sec x$; $P(\pi/3, 2)$

13 $x = t - 1$, $y = \sqrt{t}$; $P(3, 2)$

14 $x = t + 1$, $y = t^2 + 4t + 3$; $P(1, 3)$

15 $x = t - t^2$, $y = 1 - t^3$; $P(0, 1)$

16 $x = t - \sin t$, $y = 1 - \cos t$; $P(\pi/2 - 1, 1)$

17 $x = 2\sin t$, $y = 3\cos t$; $P(1, \frac{3}{2}\sqrt{3})$

18 $x = \cos^3 t$, $y = \sin^3 t$; $P(\frac{1}{4}\sqrt{2}, \frac{1}{4}\sqrt{2})$

Exer. 19–22: For the given curve and point *P*, (a) find the radius of curvature, (b) find the center of curvature, and (c) sketch the graph and the circle of curvature for *P*.

19 $y = \sin x$; $P(\pi/2, 1)$

20 $y = \sec x$; $P(0, 1)$

21 $y = e^x$; $P(0, 1)$

22 $xy = 1$; $P(1, 1)$

c **Exer. 23–26: For the given curve *C* and point *P*, approximate the curvature of *C* at *P* and plot the curve *C* along with the circle of curvature for *C* at *P*.**

23 $x = \cos t$, $y = \sin(0.8t)$; $P\left(\cos\left(\frac{\pi}{3}\right), \sin\left(\frac{4\pi}{15}\right)\right)$

24 $x = 1 + \cos t$, $y = \sin(1.2t)$; $P(1 + \cos 1, \sin 1.2)$

25 $x = 5t^2$, $y = 8t - 7t^2$; $P(\frac{5}{4}, \frac{9}{4})$

26 $x = 9t^2(3 - 2t)$, $y = 12t(3 - 3t + 2t^2)$; $P(0.936, 5.952)$, $t = 0.2$

Exer. 27–32: Find the points on the given curve at which the curvature is a maximum.

27 $y = e^{-x}$

28 $y = \cosh x$

29 $9x^2 + 4y^2 = 36$

30 $9x^2 - 4y^2 = 36$

31 $y = \ln x$

32 $y = \sin x$

Exer. 33–36: Find the points on the graph of the equation at which the curvature is 0.

33 $y = x^4 - 12x^2$

34 $y = \tan x$

35 $y = \sinh x$

36 $y = e^{-x^2}$

37 Suppose that a curve C is the graph of a polar equation $r = f(\theta)$. If $r' = dr/d\theta$ and $r'' = d^2r/d\theta^2$, show that the curvature K at $P(r, \theta)$ is

$$K = \frac{\left|2(r')^2 - rr'' + r^2\right|}{[(r')^2 + r^2]^{3/2}}.$$

(*Hint:* Use $x = r\cos\theta$ and $y = r\sin\theta$ to express C in parametric form.)

Exer. 38–40: Use the formula in Exercise 37 to find the curvature of the polar curve at $P(r, \theta)$.

38 $r = a(1 - \cos\theta)$; $0 < \theta < 2\pi$

39 $r = \sin 2\theta$; $0 < \theta < 2\pi$

40 $r = e^{a\theta}$

41 Let $P(x, y)$ be a point on the graph of $y = f(x)$ at which $K \neq 0$. If (h, k) is the center of curvature for P, show that

$$h = x - \frac{y'[1 + (y')^2]}{y''}, \qquad k = y + \frac{[1 + (y')^2]}{y''}.$$

Exer. 42–46: Use the formulas in Exercise 41 to find the center of curvature for the point P on the graph of the equation. (Refer to Exercises 7–11.)

42 $y = 2 - x^3$; $P(1, 1)$

43 $y = x^4$; $P(1, 1)$

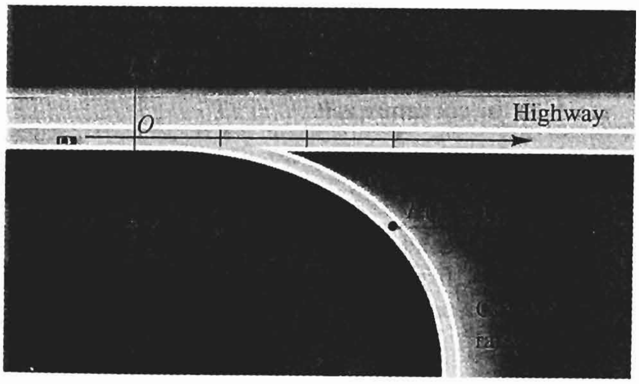
44 $y = e^{(x^2)}$; $P(0, 1)$

45 $y = \ln(x - 1)$; $P(2, 0)$

46 $y = \cos 2x$; $P(0, 1)$

47 The path of a highway and exit ramp are superimposed on a rectangular coordinate system such that the highway coincides with the x-axis. The exit ramp begins at the origin O. After following the graph of $y = -\frac{1}{27}x^3$ from O to the point $P(3, -1)$, the path follows along the arc of a circle, as shown in the figure. If $K(x)$ is the curvature of the exit ramp at (x, y), find the center of the circular arc that makes the curvature at $P(3, -1)$ continuous at $x = 3$.

Exercise 47



48 Use the equation $y = mx + b$ to prove that the curvature at every point on a line is 0 (see Example 4).

49 Prove that the maximum curvature of a parabola is at the vertex.

50 Prove that the maximum and minimum curvatures of an ellipse are at the ends of the major and minor axes, respectively.

51 Prove that the maximum curvature of a hyperbola is at the ends of the transverse axis.

52 Prove that lines and circles are the only *plane* curves that have a constant curvature. (Exercise 17 of Section 11.5 shows that this result is not true for space curves.)

Exer. 53–56: If the curve C in Figure 11.23 has a smooth parametrization $x = f(t)$, $y = g(t)$, then, by Theorem (9.6), the relationship between t and the arc length parameter s is given by

$$s = \int_a^t \sqrt{[f'(u)]^2 + [g'(u)]^2}\, du,$$

where a is the value of t corresponding to the fixed point A. Use this relationship to express the given curve in terms of the arc length parameter s if the fixed point A corresponds to $t = 0$. (*Hint:* First evaluate the integral to find the relationship between t and s, and then substitute for t in the parametric equations.)

53 $x = 4t - 3$, $y = 3t + 5$; $t \geq 0$

54 $x = 3t^2$, $y = 2t^3$; $t \geq 0$

55 $x = 4\cos t$, $y = 4\sin t$; $0 \leq t \leq 2\pi$

56 $x = e^t \cos t$, $y = e^t \sin t$; $t \geq 0$

57 Prove that if k is a nonnegative function that is nonzero and continuous on an interval $[0, a]$, then there is a plane curve C such that k represents the curvature of C as a function of arc length. (*Hint:* If s is in $[0, a]$, define

$$h(s) = \int_0^s k(t)\, dt, \qquad x = f(s) = \int_0^s \cos h(t)\, dt,$$

$$\text{and} \quad y = g(s) = \int_0^s \sin h(t)\, dt.)$$

58 Use Exercise 57 to work Exercise 52.

SOLUTION

(a) If we let

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k},$$

then the curve is the same as that considered in Example 1. Substituting the expressions obtained there for $\mathbf{r}'(t)$ and $\mathbf{r}'(t) \times \mathbf{r}''(t)$ into the formula for K in Theorem (11.25) yields

$$K = \frac{2(9t^4 + 9t^2 + 1)^{1/2}}{(9t^4 + 4t^2 + 1)^{3/2}}.$$

We could also find K by substituting for $a_\mathbf{N}$ and $\|\mathbf{r}'(t)\|$ in Theorem (11.25).

(b) Substituting $t = 1, 2, 3,$ and 4 into the formula for K obtained in part (a), we obtain the following approximations for K (compare with the table on page 962).

t	1	2	3	4
(x, y, z)	(1, 1, 1)	(2, 4, 8)	(3, 9, 27)	(4, 16, 64)
K	0.1664	0.0132	0.0027	0.0009

Using limit theorems, we can show that $\lim_{t\to\infty} K = 0$—that is, the curvature of the curve approaches that of a line as t increases.

EXERCISES 11.5

Exer. 1–8: Find general formulas for the tangential and normal components of acceleration and for the curvature of the curve C determined by $\mathbf{r}(t)$.

1 $\mathbf{r}(t) = t^2\mathbf{i} + (3t + 2)\mathbf{j}$ **2** $\mathbf{r}(t) = (2t^2 - 1)\mathbf{i} + 5t\mathbf{j}$

3 $\mathbf{r}(t) = 3t\mathbf{i} + t^3\mathbf{j} + 3t^2\mathbf{k}$ **4** $\mathbf{r}(t) = 4t\mathbf{i} + t^2\mathbf{j} + 2t^2\mathbf{k}$

5 $\mathbf{r}(t) = t(\cos t\mathbf{i} + \sin t\mathbf{j})$ **6** $\mathbf{r}(t) = \cosh t\mathbf{i} + \sinh t\mathbf{j}$

7 $\mathbf{r}(t) = 4\cos t\mathbf{i} + 9\sin t\mathbf{j} + t\mathbf{k}$

8 $\mathbf{r}(t) = e^t(\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k})$

9 A point moves along the parabola $y = x^2$ such that the horizontal component of velocity is always 3. Find the tangential and normal components of acceleration at $P(1, 1)$.

10 Work Exercise 9 if the point moves along the graph of $y = 2x^3 - x$.

11 Prove that if a point moves along a curve C with a constant speed, then the acceleration is always normal to C.

12 Use Theorem (11.25) to prove that if a point moves through space with an acceleration that is always **0**, then the motion is on a line.

13 If a point P moves along a curve C with a constant speed, show that the magnitude of the acceleration is directly proportional to the curvature of the curve.

14 If, in Exercise 13, a second point Q moves along C with a speed twice that of P, show that the magnitude of the acceleration of Q is four times greater than that of P.

15 Show that if a point moves along the graph of $y = f(x)$ for $a \le x \le b$, then the normal component of acceleration is 0 at a point of inflection.

16 If a plane curve is given parametrically by $x = f(t)$, $y = g(t)$ and if f'' and g'' exist, use Theorem (11.25) to prove that the curvature at the point $P(x, y)$ is given by Theorem (11.19).

17 Show that the curvature at every point on the circular helix $x = a\cos t$, $y = a\sin t$, $z = bt$, where $a > 0$, is given by $K = a/(a^2 + b^2)$.

18 An *elliptic helix* has parametric equations $x = a\cos t$, $y = b\sin t$, $z = ct$, where a, b, and c are positive real numbers and $a \neq b$. Find the curvature at (x, y, z).

11.6 KEPLER'S LAWS

It is fitting to conclude this chapter with a display of the power and beauty of vector methods when applied to the derivation of three classical physical laws. The discussion in this section is not simple, for it is not a simple problem that we intend to consider. There is no exercise set at the end of this section. The reason is that we are not interested in numerical calculations involving the laws to be developed. Your objective should be to carefully read and understand each step of the discussion. Proceed slowly. You will gain considerable insight into vector methods by studying the material that follows.

After many years of analyzing an enormous amount of empirical data, the German astronomer Johannes Kepler (1571–1630) formulated three laws that describe the motion of planets about the sun. These laws may be stated as follows.

***Kepler's Laws* 11.26**

First Law: The orbit of each planet is an ellipse with the sun at one focus.

Second Law: The vector from the sun to a moving planet sweeps out area at a constant rate.

Third Law: If the time required for a planet to travel once around its elliptical orbit is T and if the major axis of the ellipse is $2a$, then $T^2 = ka^3$ for some constant k.

Approximately 50 years later, Sir Isaac Newton (see *Mathematicians and Their Times*, Chapter 3) proved that Kepler's laws were consequences of Newton's law of universal gravitation and second law of motion. The achievements of both men were monumental, because these laws clarified all astronomical observations that had been made up to that time.

In this section, we shall prove Kepler's laws through the use of vectors. Since the force of gravity that the sun exerts on a planet far exceeds that exerted by other celestial bodies, we shall neglect all other forces acting on a planet. From this point of view, we have only two objects to consider: the sun and a planet revolving around it.

Theorem 12.7

If a function f of two variables is continuous at (a, b) and a function g of one variable is continuous at $f(a, b)$, then the function h defined by $h(x, y) = g(f(x, y))$ is continuous at (a, b).

Theorem (12.7) allows us to establish the continuity of composite functions of several variables, as illustrated in the next example.

EXAMPLE 7 If $h(x, y) = e^{x^2+5xy+y^3}$, show that h is continuous at every pair (a, b).

SOLUTION If we let $f(x, y) = x^2 + 5xy + y^3$ and $g(t) = e^t$, it follows that $h(x, y) = g(f(x, y))$. Since f is a polynomial function, it is continuous at every pair (a, b). Moreover, g is continuous at every $t = f(a, b)$. Thus, by Theorem (12.7), h is continuous at (a, b).

EXERCISES 12.2

Exer. 1–10: Find the limit.

1 $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - 2}{3 + xy}$

2 $\lim\limits_{(x,y)\to(2,1)} \dfrac{4 + x}{2 - y}$

3 $\lim\limits_{(x,y)\to(\pi/2,1)} \dfrac{y + 1}{2 - \cos x}$

4 $\lim\limits_{(x,y)\to(-1,3)} \dfrac{y^2 + x}{(x - 1)(y + 2)}$

5 $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4 - y^4}{x^2 + y^2}$

6 $\lim\limits_{(x,y)\to(1,0)} \dfrac{x^4 - 2x^3 + x^2 + x^2y^2}{(x - 1)^2 + y^2}$

7 $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^3 - 2x^2y + 3y^2x - 2y^3}{x^2 + y^2}$

8 $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3 - x^2y + xy^2 - y^3}{x^2 + y^2}$

9 $\lim\limits_{(x,y,z)\to(0,2,0)} \dfrac{x^2y^2 + y^4 - 4y^3 + 4y^2 + y^2z^2}{x^2 + (y - 2)^2 + z^2}$

10 $\lim\limits_{(x,y,z)\to(0,2,-3)} \dfrac{x^4 - (z + 3)^4}{x^2 + (z + 3)^2}$

Exer. 11–20: Show that the limit does not exist.

11 $\lim\limits_{(x,y)\to(0,0)} \dfrac{2x^2 - y^2}{x^2 + 2y^2}$

12 $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - 2xy + 5y^2}{3x^2 + 4y^2}$

13 $\lim\limits_{(x,y)\to(1,2)} \dfrac{xy - 2x - y + 2}{x^2 + y^2 - 2x - 4y + 5}$

14 $\lim\limits_{(x,y)\to(-2,1)} \dfrac{xy - x + 2y - 2}{(x + 2)^2 + (y - 1)^2}$

15 $\lim\limits_{(x,y)\to(0,0)} \dfrac{4x^3y}{2x^4 + 3y^4}$

16 $\lim\limits_{(x,y)\to(0,0)} \dfrac{3xy}{5x^4 + 2y^4}$

17 $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{xy + yz + xz}{x^2 + y^2 + z^2}$

18 $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{2x^2 + 3y^2 + z^2}{x^2 + y^2 + z^2}$

19 $\lim\limits_{(x,y,z)\to(-3,0,0)} \dfrac{(x - z + 3)^2}{(x + 3)^2 + y^2 + z^2}$

20 $\lim\limits_{(x,y,z)\to(2,0,0)} \dfrac{(x - 2)yz^2}{(x - 2)^4 + y^4}$

Exer. 21–24: Use polar coordinates to find the limit, if it exists.

21 $\lim_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^2}$

22 $\lim_{(x,y)\to(0,0)} \dfrac{x^3-y^3}{x^2+y^2}$

23 $\lim_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{\sin(x^2+y^2)}$

24 $\lim_{(x,y)\to(0,0)} \dfrac{\sinh(x^2+y^2)}{x^2+y^2}$

Exer. 25–28: Describe the set of all points in the *xy*-plane at which *f* is continuous.

25 $f(x, y) = \ln(x + y - 1)$

26 $f(x, y) = \dfrac{xy}{x^2 - y^2}$

27 $f(x, y) = \sqrt{x}e^{\sqrt{1-y^2}}$

28 $f(x, y) = \sqrt{25 - x^2 - y^2}$

Exer. 29–32: Describe the set of all points in an *xyz*-coordinate system at which *f* is continuous.

29 $f(x, y, z) = \dfrac{1}{x^2 + y^2 - z^2}$

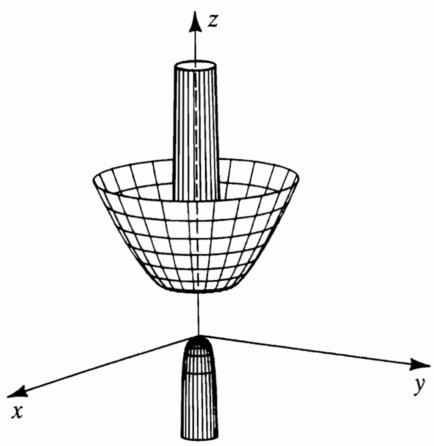
30 $f(x, y, z) = \sqrt{xy}\tan z$

31 $f(x, y, z) = \sqrt{x-2}\ln(yz)$

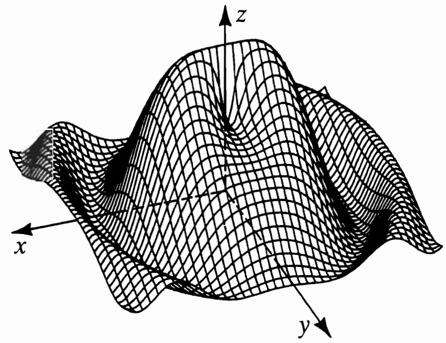
32 $f(x, y, z) = \sqrt{4 - x^2 - y^2 - z^2}$

Exer. 33–34: The graph of *f* is shown in the figure. Use polar coordinates to investigate $\lim_{(x,y)\to(0,0)} f(x, y)$.

33 $f(x, y) = \dfrac{x^2+y^2}{\ln(x^2+y^2)}$



34 $f(x, y) = \dfrac{\sin(2x^2+y^2)}{x^2+y^2}$



Exer. 35–38: Find $h(x, y) = g(f(x, y))$ and use Theorem (12.7) to determine where *h* is continuous.

35 $f(x, y) = x^2 - y^2$; $\quad g(t) = (t^2 - 4)/t$

36 $f(x, y) = 3x + 2y - 4$; $\quad g(t) = \ln(t + 5)$

37 $f(x, y) = x + \tan y$; $\quad g(z) = z^2 + 1$

38 $f(x, y) = y \ln x$; $\quad g(w) = e^w$

39 If $f(x, y) = x^2 + 2y$, $g(t) = e^t$, and $h(t) = t^2 - 3t$, find $g(f(x, y))$, $h(f(x, y))$, and $f(g(t), h(t))$.

40 If $f(x, y, z) = 2x + ye^z$ and $g(t) = t^2$, find $g(f(x, y, z))$.

41 If $f(u, v) = uv - 3u + v$, $g(x, y) = x - 2y$, and $k(x, y) = 2x + y$, find $f(g(x, y), k(x, y))$.

42 If $f(x, y) = 2x + y$, find $f(f(x, y), f(x, y))$.

43 Extend Definition (12.3) to functions of four variables.

44 Prove that if f is a continuous function of two variables and $f(a, b) > 0$, then there is a circle C in the xy-plane with center (a, b) such that $f(x, y) > 0$ for every pair (x, y) that is in the domain of f and within C.

45 Prove, directly from Definition (12.3), that

(a) $\lim_{(x,y)\to(a,b)} x = a$ **(b)** $\lim_{(x,y)\to(a,b)} y = b$

46 If $\lim_{(x,y)\to(a,b)} f(x, y) = L$ and c is any real number, prove, directly from Definition (12.3), that

$$\lim_{(x,y)\to(a,b)} cf(x, y) = cL.$$

Third and higher partial derivatives are defined in similar fashion. For example,

$$\frac{\partial}{\partial x} f_{xx} = f_{xxx} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial^3 f}{\partial x^3},$$

$$\frac{\partial}{\partial x} f_{xy} = f_{xyx} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y\,\partial x}\right) = \frac{\partial^3 f}{\partial x\,\partial y\,\partial x},$$

and so on. If first, second, and third partial derivatives are continuous, then the order of differentiation is immaterial—that is,

$$f_{xyx} = f_{yxx} = f_{xxy} \quad \text{and} \quad f_{yxy} = f_{xyy} = f_{yyx}.$$

Of course, letters other than x and y may be used. If f is a function of r and s, then symbols such as

$$f_r(r, s), \qquad f_s(r, s), \qquad f_{rs}, \qquad \frac{\partial f}{\partial r}, \qquad \frac{\partial^2 f}{\partial r^2}$$

are used for partial derivatives.

Similar notations and results apply to partials of functions of more than two variables.

EXERCISES 12.3

Exer. 1–18: Find the first partial derivatives of f.

1 $f(x, y) = 2x^4y^3 - xy^2 + 3y + 1$

2 $f(x, y) = (x^3 - y^2)^5$

3 $f(r, s) = \sqrt{r^2 + s^2}$

4 $f(s, t) = \dfrac{t}{s} - \dfrac{s}{t}$

5 $f(x, y) = xe^y + y \sin x$

6 $f(x, y) = e^x \ln xy$

7 $f(t, v) = \ln\sqrt{\dfrac{t + v}{t - v}}$

8 $f(u, w) = \arctan \dfrac{u}{w}$

9 $f(x, y) = x \cos \dfrac{x}{y}$

10 $f(x, y) = \sqrt{4x^2 - y^2} \sec x$

11 $f(x, y, z) = 3x^2z + xy^2$

12 $f(x, y, z) = x^2y^3z^4 + 2x - 5yz$

13 $f(r, s, t) = r^2e^{2s} \cos t$

14 $f(x, y, t) = \dfrac{x^2 - t^2}{1 + \sin 3y}$

15 $f(x, y, z) = xe^z - ye^x + ze^{-y}$

16 $f(r, s, v, p) = r^3 \tan s + \sqrt{s}e^{(v^2)} - v \cos 2p$

17 $f(q, v, w) = \sin^{-1}\sqrt{qv} + \sin vw$

18 $f(x, y, z) = xyze^{xyz}$

Exer. 19–24: Verify that $w_{xy} = w_{yx}$.

19 $w = xy^4 - 2x^2y^3 + 4x^2 - 3y$

20 $w = \dfrac{x^2}{x + y}$

21 $w = x^3e^{-2y} + y^{-2} \cos x$

22 $w = y^2e^{(x^2)} + \dfrac{1}{x^2y^3}$

23 $w = x^2 \cosh \dfrac{z}{y}$

24 $w = \sqrt{x^2 + y^2 + z^2}$

25 If $w = 3x^2y^3z + 2xy^4z^2 - yz$, find w_{xyz}.

26 If $w = u^4vt^2 - 3uv^2t^3$, find w_{tut}.

27 If $u = v \sec rt$, find u_{rvr}.

28 If $v = y \ln(x^2 + z^4)$, find v_{zzy}.

29 If $w = \sin xyz$, find $\dfrac{\partial^3 w}{\partial z\,\partial y\,\partial x}$.

30 If $w = \dfrac{x^2}{y^2 + z^2}$, find $\dfrac{\partial^3 w}{\partial z\,\partial y^2}$.

31 If $w = r^4s^3t - 3s^2e^{rt}$, verify that $w_{rrs} = w_{rsr} = w_{srr}$.

32 If $w = \tan uv + 2\ln(u + v)$, verify that $w_{uvv} = w_{vuv} = w_{vvu}$.

Exer. 33–36: A function f of x and y is *harmonic* if

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

throughout the domain of f. Prove that the given function is harmonic.

33 $f(x, y) = \ln\sqrt{x^2 + y^2}$

34 $f(x, y) = \arctan \dfrac{y}{x}$

35 $f(x, y) = \cos x \sinh y + \sin x \cosh y$

36 $f(x, y) = e^{-x}\cos y + e^{-y}\cos x$

37 If $w = \cos(x - y) + \ln(x + y)$, show that

$$\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} = 0$$

38 If $w = (y - 2x)^3 - \sqrt{y - 2x}$, show that $w_{xx} - 4w_{yy} = 0$.

39 If $w = e^{-c^2t}\sin cx$, show that $w_{xx} = w_t$ for every real number c.

40 The ideal gas law may be stated as $PV = knT$, where n is the number of moles of gas, V is the volume, T is the temperature, P is the pressure, and k is a constant. Show that

$$\frac{\partial V}{\partial T}\frac{\partial T}{\partial P}\frac{\partial P}{\partial V} = -1.$$

Exer. 41–42: Show that v satisfies the *wave equation*

$$\frac{\partial^2 v}{\partial t^2} = a^2\frac{\partial^2 v}{\partial x^2}.$$

41 $v = (\sin akt)(\sin kx)$

42 $v = (x - at)^4 + \cos(x + at)$

Exer. 43–46: Show that the functions u and v satisfy the *Cauchy–Riemann equations* $u_x = v_y$ and $u_y = -v_x$.

43 $u(x, y) = x^2 - y^2$; $v(x, y) = 2xy$

44 $u(x, y) = \dfrac{y}{x^2 + y^2}$; $v(x, y) = \dfrac{x}{x^2 + y^2}$

45 $u(x, y) = e^x \cos y$; $v(x, y) = e^x \sin y$

46 $u(x, y) = \cos x \cosh y + \sin x \sinh y$;
$v(x, y) = \cos x \cosh y - \sin x \sinh y$

47 List all possible second partial derivatives of $w = f(x, y, z)$.

48 If $w = f(x, y, z, t, v)$, define w_t as a limit.

49 A flat metal plate lies on an xy-plane such that the temperature T at (x, y) is given by $T = 10(x^2 + y^2)^2$, where T is in degrees and x and y are in centimeters. Find the instantaneous rate of change of T with respect to distance at $(1, 2)$ in the direction of

(a) the x-axis

(b) the y-axis

50 The surface of a certain lake is represented by a region D in an xy-plane such that the depth under the point corresponding to (x, y) is given by the function $f(x, y) = 300 - 2x^2 - 3y^2$, where x, y, and $f(x, y)$ are in feet. If a water skier is in the water at the point $(4, 9)$, find the instantaneous rate at which the depth changes in the direction of

(a) the x-axis

(b) the y-axis

51 Suppose the electrical potential V at the point (x, y, z) is given by $V = 100/(x^2 + y^2 + z^2)$, where V is in volts and x, y, and z are in inches. Find the instantaneous rate of change of V with respect to distance at $(2, -1, 1)$ in the direction of

(a) the x-axis

(b) the y-axis

(c) the z-axis

52 An object is situated in a rectangular coordinate system such that the temperature T at the point $P(x, y, z)$ is given by $T = 4x^2 - y^2 + 16z^2$, where T is in degrees and x, y, and z are in centimeters. Find the instantaneous rate of change of T with respect to distance at the point $P(4, -2, 1)$ in the direction of

(a) the x-axis

(b) the y-axis

(c) the z-axis

53 When a pollutant such as nitric oxide is emitted from a smokestack of height h meters, the long-range concentration $C(x, y)$ (in $\mu g/m^3$) of the pollutant at a point x kilometers from the smokestack and at a height of y meters (see figure) can often be represented by

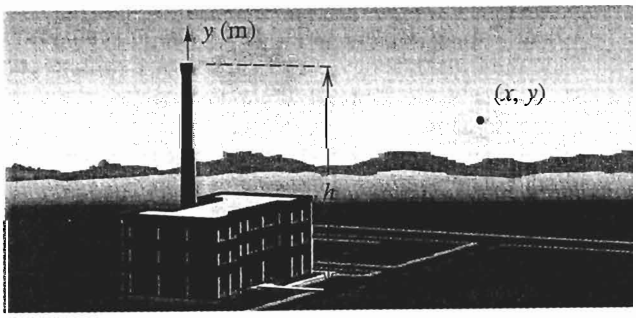
$$C(x, y) = \frac{a}{x^2}[e^{-b(y-h)^2/x^2} + e^{-b(y+h)^2/x^2}],$$

where a and b are positive constants that depend on atmospheric conditions and the pollution emission rate. Suppose that

$$C(x, y) = \frac{200}{x^2}[e^{-0.02(y-10)^2/x^2} + e^{-0.02(y+10)^2/x^2}].$$

Compute and interpret $\partial C/\partial x$ and $\partial C/\partial y$ at the point (2, 5).

Exercise 53



54 The analysis of certain electrical circuits involves the formula $I = V/\sqrt{R^2 + L^2\omega^2}$, where I is the current, V the voltage, R the resistance, L the inductance, and ω a positive constant. Find and interpret $\partial I/\partial R$ and $\partial I/\partial L$.

55 Most computers have only one processor that can be used for computations. Modern supercomputers, however, have anywhere from two to several thousand processors. A multiprocessor supercomputer is compared with a uniprocessor computer in terms of speedup. The *speedup* S is the number of times faster that a given computation can be accomplished with a multiprocessor than with a uniprocessor. A formula used to determine S is *Amdahl's law*,

$$S(p, q) = \frac{p}{q + p(1-q)},$$

where p is the number of processors and q is the fraction of the computation that can be performed using all available processors in parallel—that is, using them in such a way that data are processed concurrently by separate units. The ideal situation, *complete parallelism*, occurs when $q = 1$.

(a) If $q = 0.8$, find the speedup when $p = 10$, 100, and 1000. Show that the speedup S cannot exceed 5, regardless of the number of processors available.

(b) Find the instantaneous rate of change of S with respect to q.

(c) What is the rate of change in part (b) if there is complete parallelism, and how does the number of processors affect this rate of change?

56 Refer to Exercise 55. The efficiency E of a multiprocessor computation can be calculated using the equation

$$E = \frac{S(p, q)}{p} = \frac{1}{q + p(1-q)}.$$

Show that if $0 \le q < 1$, E is a decreasing function of p and therefore, without complete parallelism, increasing the number of processors does not increase the efficiency of the computation.

57 In the study of frost penetration in highway engineering, the temperature T at time t hours and depth x feet can be approximated by

$$T = T_0 e^{-\lambda x} \sin(\omega t - \lambda x),$$

where T_0, ω, and λ are constants. Assume that the period of $\sin(\omega t - \lambda x)$ is 24 hr.

(a) Find and interpret $\partial T/\partial t$ and $\partial T/\partial x$.

(b) Show that T satisfies the one-dimensional heat equation

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2},$$

where k is a constant.

58 Show that any function given by

$$w = (\sin ax)(\cos by)e^{-\sqrt{a^2+b^2}\,z}$$

satisfies Laplace's equation in three dimensions:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0$$

59 The vital capacity V of the lungs is the largest volume (in milliliters) that can be exhaled after a maximum inhalation of air. For a typical male x years old and y centimeters tall, V may be approximated by the formula $V = 27.63y - 0.112xy$. Compute and interpret

(a) $\partial V/\partial x$ **(b)** $\partial V/\partial y$

60 On a clear day, the intensity of sunlight $I(x, t)$ (in foot-candles) at t hours after sunrise and at ocean depth x (in meters) can be approximated by

$$I(x, t) = I_0 e^{-kx} \sin^3(\pi t/D),$$

where I_0 is the intensity at midday, D is the length of the day (in hours), and k is a positive constant. If $I_0 = 1000$, $D = 12$, and $k = 0.10$, compute and interpret $\partial I/\partial t$ and $\partial I/\partial x$ when $t = 6$ and $x = 5$.

61 In economics, the *price elasticity of demand* for a commodity indicates the responsiveness of consumers to a change in the market price of the commodity. Suppose n commodities $C_1, C_2, \ldots, C_n$ have prices $p_1, p_2, \ldots, p_n$, respectively, and consumer demand for C_k is a function q_k of $p_1, p_2, \ldots, p_n$. The price elasticity of C_k is the function e_k defined by

$$e_k = \frac{p_k}{q_k}\frac{\partial q_k}{\partial p_k}.$$

If, for each k,

$$q_k = b_k p_1^{-a_{k1}} p_2^{-a_{k2}} \cdots p_n^{-a_{kn}},$$

where $a_{k1}, a_{k2}, \ldots, a_{kn}$ and b_k are nonnegative constants, show that e_k is a constant function.

62 Refer to Exercise 61. Commodity C_k is said to be *independent* of commodity C_j if a change in price p_k does not affect demand q_j. This is equivalent to the condition $\partial q_j/\partial p_k = 0$. If q_k has the form in Exercise 61, show that C_k is independent of C_j if and only if $a_{jk} = 0$.

Exer. 63–64: Use Theorem (12.10).

63 Let C be the trace of the paraboloid $z = 9 - x^2 - y^2$ on the plane $x = 1$. Find parametric equations of the tangent line l to C at the point $P(1, 2, 4)$. Sketch the paraboloid, C, and l.

64 Let C be the trace of the graph of the equation $z = \sqrt{36 - 9x^2 - 4y^2}$ on the plane $y = 2$. Find parametric equations of the tangent line l to C at the point $P(1, 2, \sqrt{11})$. Sketch the surface, C, and l.

c **Exer. 65–68: Use the following formulas with $h = 0.01$ to approximate $f_x(0.5, 0.2)$ and $f_y(0.5, 0.2)$, and compare the results with the values obtained from $f_x(x, y)$ and $f_y(x, y)$.**

$$f_x(x, y) \approx \frac{f(x+h, y) - f(x-h, y)}{2h}$$

$$f_y(x, y) \approx \frac{f(x, y+h) - f(x, y-h)}{2h}$$

65 $f(x, y) = y^2 \sin(xy)$

66 $f(x, y) = xy^3 + 4x^3y^2$

67 $f(x, y) = \dfrac{x^2y^2}{3x^2 + y^2}$

68 $f(x, y) = e^{(1-2x^2-y^2)}$

c **Exer. 69–70: Use the following formulas with $h = 0.01$ to approximate $f_{xx}(0.6, 0.8)$ and $f_{yy}(0.6, 0.8)$.**

$$f_{xx}(x, y) \approx \frac{f(x+h, y) - 2f(x, y) + f(x-h, y)}{h^2}$$

$$f_{yy}(x, y) \approx \frac{f(x, y+h) - 2f(x, y) + f(x, y-h)}{h^2}$$

69 $f(x, y) = \sec^2[\tan(xy^2)]\sin(xy)$

70 $f(x, y) = \dfrac{x^3 + xy^2}{\tan(xy) + 4x^2y^3}$

12.4 INCREMENTS AND DIFFERENTIALS

We continue, in this section, to consider differentiation of real-valued functions of two or three variables. We introduce *increments* and *differentials* and develop a definition of *differentiability* for functions of two variables.

If f is a function of two variables x and y, then the symbols Δx and Δy denote increments of x and y. Note that Δy is an increment of the *independent* variable y and is not the same as that defined in (2.30) for a *dependent*

$f_x(x_k, y_k)$	$f_y(x_k, y_k)$	$g_x(x_k, y_k)$	$g_y(x_k, y_k)$	Δx_k	Δy_k
1	1	−0.87758256	1	0.27725783	0.22274217
1.55451565	1.44548435	−0.71283938	1	−0.03695719	−0.04775939
1.48060127	1.34996558	−0.73826581	1	−0.00121442	−0.00136946
1.47817243	1.34722665	−0.73908440	1	−0.00000108	−0.00000130
1.47817027	1.34722406	−0.73908513	1	0.00000000	0.00000000

From the table,

$$(x_5, y_5) \approx (0.73908513, 0.67361203).$$

An approximation to the solution corresponding to the point in the third quadrant in Figure 12.37 is, by symmetry, $(-x_5, -y_5)$.

EXERCISES 12.4

Exer. 1–2: If Δx and Δy are increments of x and y, find (a) Δw, (b) dw, and (c) $dw - \Delta w$.

1 $w = 5y^2 - xy$ **2** $w = xy - y^2 + 3x$

Exer. 3–6: Find expressions for ϵ_1 and ϵ_2 that satisfy Definition (12.16).

3 $f(x, y) = 4y^2 - 3xy + 2x$

4 $f(x, y) = (2x - y)^2$ **5** $f(x, y) = x^3 + y^3$

6 $f(x, y) = 2x^2 - xy^2 + 3y$

Exer. 7–18: Find dw.

7 $w = x^3 - x^2y + 3y^2$ **8** $w = 5x^2 + 4y - 3xy^3$

9 $w = x^2 \sin y + 2y^{3/2}$ **10** $w = ye^{-2x} - 3x^4$

11 $w = x^2e^{xy} + (1/y^2)$

12 $w = \ln(x^2 + y^2) + x \tan^{-1} y$

13 $w = x^2 \ln(y^2 + z^2)$ **14** $w = x^2y^3z + e^{-2z}$

15 $w = \dfrac{xyz}{x + y + z}$ **16** $w = x^2e^{yz} + y \ln z$

17 $w = x^2z + 4yt^3 - xz^2t$ **18** $w = x^2y^3zt^{-1}v^4$

Exer. 19–22: Use differentials to approximate the change in f if the independent variables change as indicated.

19 $f(x, y) = x^2 - 3x^3y^2 + 4x - 2y^3 + 6$; $(-2, 3)$ to $(-2.02, 3.01)$

20 $f(x, y) = x^2 - 2xy + 3y$; $(1, 2)$ to $(1.03, 1.99)$

21 $f(x, y, z) = x^2z^3 - 3yz^2 + x^{-3} + 2y^{1/2}z$; $(1, 4, 2)$ to $(1.02, 3.97, 1.96)$

22 $f(x, y, z) = xy + xz + yz$; $(-1, 2, 3)$ to $(-0.98, 1.99, 3.03)$

23 The dimensions of a closed rectangular box are measured as 3 ft, 4 ft, and 5 ft, with a possible error of $\pm\frac{1}{16}$ in. for each measurement. Use differentials to approximate the maximum error in the calculated value of

(a) the surface area **(b)** the volume

24 The two shortest sides of a right triangle are measured as 3 cm and 4 cm, respectively, with a possible error of ± 0.02 cm in each measurement. Use differentials to approximate the maximum error in the calculated value of

(a) the hypotenuse **(b)** the area of the triangle

25 The *withdrawal resistance* of a nail indicates its holding strength in wood. An empirical formula used for bright, common nails is $P = 15{,}700S^{5/2}RD$, where P is the maximum withdrawal resistance (in pounds), S is the specific gravity of the wood at 12% moisture content, R is the radius of the nail (in inches), and D is the depth (in inches) to which the nail has penetrated the wood. A 6d bright, common nail of length 2 in. and diameter 0.113 in. is driven completely into a piece of Douglas fir that has a specific gravity of 0.54.

(a) Approximate the maximum withdrawal resistance. (In applications, only one sixth of this maximum is considered safe for extended periods of time.)

(b) When nails are manufactured, R and D can vary by $\pm 2\%$, and the specific gravity of different samples of Douglas fir can vary by $\pm 3\%$. Approximate the maximum percentage error in the calculated value of P.

26 The total resistance R of three resistances R_1, R_2, and R_3 connected in parallel is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}.$$

If measurements of R_1, R_2, and R_3 are 100, 200, and 400 ohms, respectively, with a maximum error of $\pm 1\%$ in each measurement, approximate the maximum error in the calculated value of R.

27 The specific gravity of an object more dense than water is given by $s = A/(A - W)$, where A and W are the weights (in pounds) of the object in air and water, respectively. If measurements are $A = 12$ lb and $W = 5$ lb, with maximum errors of $\pm\frac{1}{2}$ oz in air and ± 1 oz in water, what is the maximum error in the calculated value of s?

28 The pressure P, volume V, and temperature T (in °K) of a confined gas are related by the ideal gas law $PV = kT$, where k is a constant. If $P = 0.5$ lb/in^2 when $V = 64$ in^3 and $T = 350$ °K, approximate the change in P if V and T change to 70 in^3 and 345 °K, respectively.

29 Suppose that when the specific gravity formula $s = A/(A - W)$ is used (see Exercise 27), there are percentage errors of $\pm 2\%$ and $\pm 4\%$ in the measurements of A and W, respectively. Express the maximum percentage error in the calculated value of s as a function of A and W.

30 Suppose that when the ideal gas law $PV = kT$ is used (see Exercise 28), there are percentage errors of $\pm 0.8\%$ and $\pm 0.5\%$ in the measurements of T and P, respectively. Approximate the maximum percentage error in the calculated value of V.

31 The electrical resistance R of a wire is directly proportional to its length and inversely proportional to the square of its diameter. If the length is measured with a possible error of $\pm 1\%$ and the diameter is measured with a possible error of $\pm 3\%$, what is the maximum percentage error in the calculated value of R?

32 The flow of blood through an arteriole is given by $F = \pi PR^4/(8vl)$, where l is the length of the arteriole, R is the radius, P is the pressure difference between the two ends, and v is the viscosity of the blood (see Section 5.8). Suppose that v and l are constant. Use differentials to approximate the percentage change in the blood flow if the radius decreases by 2% and the pressure increases by 3%.

33 The temperature T at the point $P(x, y, z)$ in an xyz-coordinate system is given by

$$T = 8(2x^2 + 4y^2 + 9z^2)^{1/2}.$$

Use differentials to approximate the temperature difference between the points (6, 3, 2) and (6.1, 3.3, 1.98).

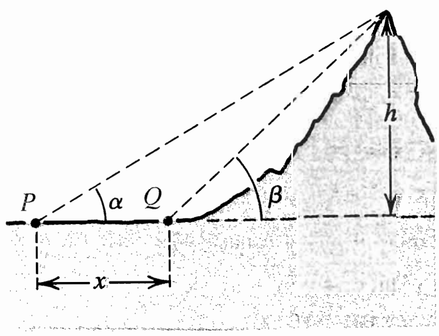
34 Approximate the change in area of an isosceles triangle if each of the two equal sides increases from 100 to 101 and the angle between them decreases from 120° to 119°.

35 If a mountaintop is viewed from the point P shown in the figure, the angle of elevation is α. From a point Q that is a distance x units closer to the mountain, the angle of elevation is β. From trigonometry, the height h of the mountain is given by

$$h = \frac{x}{\cot\alpha - \cot\beta}.$$

A surveyor measures α and β to an accuracy of 30″ (approximately 0.000145 radian). Suppose that $\alpha = 15°$, $\beta = 20°$, and $x = 2000$ ft. Use differentials to estimate, to the nearest 0.1 ft, how accurate the length measurement must be so that the maximum error in the calculated value of h is no greater than ± 10 ft.

Exercise 35



36 If a drug is taken orally, the time T at which the largest amount of drug is in the bloodstream can be calculated using the half-life x of the drug in the stomach and the half-life y of the drug in the bloodstream. For many common drugs (such as penicillin), T is given by

$$T = \frac{xy(\ln x - \ln y)}{(x - y)\ln 2}.$$

For a certain drug, $x = 30$ min and $y = 1$ hr. If the maximum error in estimating each half-life is $\pm 10\%$, find the maximum error in the calculated value of T.

37 Assume that the cylinder described in Example 4 has a closed top and bottom. Use differentials to approximate the maximum error in the calculated total surface area.

38 Use differentials to approximate the change in surface area of the box described in Example 6. What is the exact change?

Exer. 39–40: Prove that f is differentiable throughout its domain.

39 $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$

40 $f(x, y, z) = \dfrac{x + y + z}{x^2 + y^2 + z^2}$

41 Let

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

(a) Prove that $f_x(0, 0)$ and $f_y(0, 0)$ exist. (*Hint:* Use Definition (12.8).)

(b) Prove that f is not continuous at (0, 0).

(c) Prove that f is not differentiable at (0, 0).

42 Let

$$f(x, y, z) = \begin{cases} \dfrac{xyz}{x^3 + y^3 + z^3} & \text{if } (x, y, z) \neq (0, 0, 0) \\ 0 & \text{if } (x, y, z) = (0, 0, 0) \end{cases}$$

(a) Prove that f_x, f_y, and f_z exist at (0, 0, 0).

(b) Prove that f is not differentiable at (0, 0, 0).

c **Exer. 43–44: Refer to Example 7. Use Newton's method to approximate a solution of the system of equations to four decimal places using the given first approximation.**

43 $\begin{cases} \dfrac{x^2}{4} + \dfrac{y^2}{9} = 1 \\ \dfrac{(x-1)^2}{10} + \dfrac{(y+1)^2}{5} = 1 \end{cases}$ $\quad (x_1, y_1) = (2, 1)$

44 $\begin{cases} \dfrac{x^2}{9} + \dfrac{y^2}{4} = 1 \\ \dfrac{(x-1)^2}{2} - \dfrac{(y-1)^2}{3} = 1 \end{cases}$ $\quad (x_1, y_1) = (-1.5, -1.5)$

45 Derive Newton's method for a system of three equations,

$$\begin{cases} f(x, y, z) = 0 \\ g(x, y, z) = 0 \\ h(x, y, z) = 0 \end{cases}$$

where f, g, and h are functions of three variables. Express the solution in a matrix form similar to (2) on page 1017.

12.5 CHAIN RULES

If f and g are functions of *one* variable such that

$$w = f(u) \quad \text{and} \quad u = g(x),$$

then the composite function of f and g is given by

$$w = f(g(x)).$$

Applying the chain rule (2.26), we may find the derivative of w with respect to x as follows:

$$\frac{dw}{dx} = \frac{dw}{du}\frac{du}{dx}$$

In this section, we shall extend this formula to functions of several variables.

Let f, g, and h be functions of two variables such that

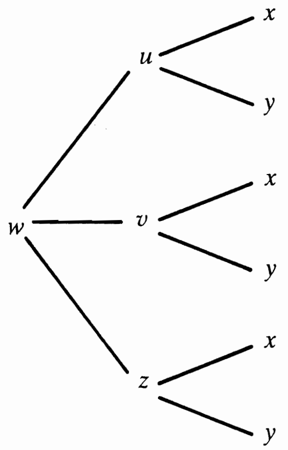
$$w = f(u, v), \quad \text{with} \quad u = g(x, y), \quad v = h(x, y).$$

PROOF The statement $F(x, y, z) = 0$ *determines a function f such that $z = f(x, y)$* means that $F(x, y, f(x, y)) = 0$ for every (x, y) in the domain of f. Consider the composite function F of x and y defined as follows:

$$w = F(u, v, z), \quad \text{where} \quad u = x, \quad v = y, \quad z = f(x, y).$$

Note that u and v are functions of x and y, since we can write u and v as $u = x + (0 \cdot y)$ and $v = y + (0 \cdot x)$. Referring to the diagram in Figure 12.46 and considering the branches that lead from w to x, we obtain

Figure 12.46



$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x}.$$

Since $w = F(x, y, f(x, y)) = 0$ for every x and every y, it follows that $\partial w/\partial x = 0$. Moreover, since $\partial u/\partial x = 1$ and $\partial v/\partial x = 0$, our chain rule formula for $\partial w/\partial x$ may be written

$$0 = \frac{\partial w}{\partial x}(1) + \frac{\partial w}{\partial y}(0) + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x},$$

and if $\partial w/\partial z \neq 0$,

$$\frac{\partial z}{\partial x} = -\frac{\partial w/\partial x}{\partial w/\partial z} = -\frac{F_x(x, y, z)}{F_z(x, y, z)}.$$

The formula for $\partial z/\partial y$ may be obtained in similar fashion. ■

EXAMPLE ■ 6 Find $\partial z/\partial x$ and $\partial z/\partial y$ if $z = f(x, y)$ is determined implicitly by

$$x^2z^2 + xy^2 - z^3 + 4yz - 5 = 0.$$

SOLUTION If we let $F(x, y, z)$ denote the expression on the left side of the given equation, then, by Theorem (12.23),

$$\frac{\partial z}{\partial x} = -\frac{2xz^2 + y^2}{2x^2z - 3z^2 + 4y}$$

$$\frac{\partial z}{\partial y} = -\frac{2xy + 4z}{2x^2z - 3z^2 + 4y}.$$

EXERCISES 12.5

Use a chain rule in Exercises 1–14.

Exer. 1–2: Find $\partial w/\partial x$ and $\partial w/\partial y$.

1 $w = u \sin v; \quad u = x^2 + y^2, \quad v = xy$

2 $w = uv + v^2; \quad u = x \sin y, \quad v = y \sin x$

Exer. 3–4: Find $\partial w/\partial r$ and $\partial w/\partial s$.

3 $w = u^2 + 2uv; \quad u = r \ln s, \quad v = 2r + s$

4 $w = e^{tv}; \quad t = r + s, \quad v = rs$

Exer. 5–6: Find $\partial z/\partial x$ and $\partial z/\partial y$.

5 $z = r^3 + s + v^2$;
$r = xe^y, \quad s = ye^x, \quad v = x^2y$

6 $z = pq + qw$;
$p = 2x - y, \quad q = x - 2y, \quad w = -2x + 2y$

Exer. 7–8: Find $\partial r/\partial u$, $\partial r/\partial v$, and $\partial r/\partial t$.

7 $r = x \ln y; \quad x = 3u + vt, \quad y = uvt$

8 $r = w^2 \cos z; \quad w = u^2vt, \quad z = ut^2$

9 If $p = u^2 + 3v^2 - 4w^2$, where $u = x - 3y + 2r - s$, $v = 2x + y - r + 2s$, and $w = -x + 2y + r + s$, find $\partial p/\partial r$.

10 Find $\partial s/\partial y$ if $s = tr + ue^v$, where $t = xy^2z$, $r = x^2yz$, $u = xyz^2$, and $v = xyz$.

Exer. 11–14: Find dw/dt.

11 $w = x^3 - y^3; \quad x = \dfrac{1}{t+1}, \quad y = \dfrac{t}{t+1}$

12 $w = \ln(u + v); \quad u = e^{-2t}, \quad v = t^3 - t^2$

13 $w = r^2 - s \tan v; \quad r = \sin^2 t, \quad s = \cos t, \quad v = 4t$

14 $w = x^2y^3z^4; \quad x = 2t + 1, \quad y = 3t - 2, \quad z = 5t + 4$

Exer. 15–18: Use partial derivatives to find dy/dx if $y = f(x)$ is determined implicitly by the given equation.

15 $2x^3 + x^2y + y^3 = 1$

16 $x^4 + 2x^2y^2 - 3xy^3 + 2x = 0$

17 $6x + \sqrt{xy} = 3y - 4$

18 $x^{2/3} + y^{2/3} = 4$

Exer. 19–22: Find $\partial z/\partial x$ and $\partial z/\partial y$ if $z = f(x, y)$ is determined implicitly by the given equation.

19 $2xz^3 - 3yz^2 + x^2y^2 + 4z = 0$

20 $xz^2 + 2x^2y - 4y^2z + 3y - 2 = 0$

21 $xe^{yz} - 2ye^{xz} + 3ze^{xy} = 1$

22 $yx^2 + z^2 + \cos xyz = 4$

Exer. 23–32: Use a chain rule.

23 The radius r and altitude h of a right circular cylinder are increasing at rates of 0.01 in./min and 0.02 in./min, respectively.

(a) Find the rate at which the volume is increasing at the time when $r = 4$ in. and $h = 7$ in.

(b) At what rate is the curved surface area changing at this time?

24 The equal sides and the included angle of an isosceles triangle are increasing at rates of 0.1 ft/hr and 2°/hr, respectively. Find the rate at which the area of the triangle is increasing at the time when the length of each of the equal sides is 20 ft and the included angle is 60°.

25 The pressure P, volume V, and temperature T of a confined gas are related by the ideal gas law $PV = kT$, where k is a constant. If P and V are changing at the rates dP/dt and dV/dt, respectively, find a formula for dT/dt.

26 If the base radius r and altitude h of a right circular cylinder are changing at the rates dr/dt and dh/dt, respectively, find a formula for dV/dt, where V is the volume of the cylinder.

27 A certain gas obeys the ideal gas law $PV = 8T$. Suppose that the gas is being heated at a rate of 2°/min and the pressure is increasing at a rate of $\frac{1}{2}$ (lb/in^2)/min. If, at a certain instant, the temperature is 200° and the pressure is 10 lb/in^2, find the rate at which the volume is changing.

28 Sand is leaking from a hole in a container at a rate of 6 in^3/min. As it leaks out, it forms a pile in the shape of a right circular cone whose base radius is increasing at a rate of $\frac{1}{4}$ in./min. If, at the instant that 40 in^3 has leaked out, the radius is 5 in., find the rate at which the height of the pile is increasing.

29 At age 2 yr, a typical boy is 86 cm tall, weighs 13 kg, and is growing at the rate of 9 cm/year and 2 kg/year. Use the DuBois and DuBois surface area formula, $S = 0.007184x^{0.425}y^{0.725}$ for weight x and height y, to estimate the rate at which the body surface area is growing (see Exercise 59 of Section 12.1).

30 When the size of the molecules and their forces of attraction are taken into account, the pressure P, volume V, and temperature T of a mole of confined gas are related by the *van der Waals equation*,

$$\left(P + \frac{a}{V^2}\right)(V - b) = kT,$$

where a, b, and k are positive constants. If t is time, find a formula for dT/dt in terms of dP/dt, dV/dt, P, and V.

31 If n resistances $R_1, R_2, \ldots, R_n$ are connected in parallel, then the total resistance R is given by

$$\frac{1}{R} = \sum_{k=1}^{n} \frac{1}{R_k}.$$

Prove that for $k = 1, 2, \ldots, n$,

$$\frac{\partial R}{\partial R_k} = \left(\frac{R}{R_k}\right)^2.$$

32 A function f of two variables is *homogeneous of degree* n if $f(tx, ty) = t^n f(x, y)$ for every t such that (tx, ty) is in the domain of f. Show that, for such functions, $xf_x(x, y) + yf_y(x, y) = nf(x, y)$. (*Hint:* Differentiate $f(tx, ty)$ with respect to t.)

Exer. 33–36: Refer to Exercise 32. Find the degree n of the homogeneous function f and verify the formula

$$xf_x(x, y) + yf_y(x, y) = nf(x, y).$$

33 $f(x, y) = 2x^3 + 3x^2y + y^3$

34 $f(x, y) = \dfrac{x^3y}{x^2 + y^2}$

35 $f(x, y) = \arctan \dfrac{y}{x}$

36 $f(x, y) = xye^{y/x}$

37 If $w = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, show that

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2.$$

38 If $w = f(x, y)$, where $x = e^r\cos\theta$ and $y = e^r\sin\theta$, show that

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = e^{-2r}\left(\frac{\partial^2 w}{\partial r^2} + \frac{\partial^2 w}{\partial \theta^2}\right).$$

39 If $w = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, show that

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r}\frac{\partial w}{\partial r}.$$

40 If $v = f(x - at) + g(x + at)$ and f and g have second partial derivatives, show that v satisfies the wave equation

$$\frac{\partial^2 v}{\partial t^2} = a^2\frac{\partial^2 v}{\partial x^2}.$$

(Compare with Exercise 42 of Section 12.3.)

41 If $w = \cos(x + y) + \cos(x - y)$, show, without using addition formulas, that $w_{xx} - w_{yy} = 0$.

42 If $w = f(x^2 + y^2)$, show that $y(\partial w/\partial x) - x(\partial w/\partial y) = 0$. (*Hint:* Let $u = x^2 + y^2$.)

43 If $w = f(u, v)$, where $u = g(x, y)$ and $v = k(x, y)$, show that

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial^2 w}{\partial v\,\partial u} + \frac{\partial^2 w}{\partial u\,\partial v}\right)\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}$$
$$+ \frac{\partial^2 w}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial w}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial w}{\partial v}\frac{\partial^2 v}{\partial x^2}.$$

44 For w, u, and v as given in Exercise 43, show that

$$\frac{\partial^2 w}{\partial y\,\partial x} = \frac{\partial^2 w}{\partial u^2}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial^2 w}{\partial v\,\partial u}\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial^2 w}{\partial u\,\partial v}\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$
$$+ \frac{\partial^2 w}{\partial v^2}\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial u}\frac{\partial^2 u}{\partial y\,\partial x} + \frac{\partial w}{\partial v}\frac{\partial^2 v}{\partial y\,\partial x}.$$

45 Suppose that $u = f(x, y)$ and $v = g(x, y)$ satisfy the Cauchy–Riemann equations $u_x = v_y$ and $u_y = -v_x$. If $x = r\cos\theta$ and $y = r\sin\theta$, show that

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

46 If $u = y/(x^2 + y^2)$ and $v = x/(x^2 + y^2)$, verify the formulas for $\partial u/\partial r$ and $\partial v/\partial r$ in Exercise 45 directly, by substituting $r\cos\theta$ for x and $r\sin\theta$ for y and then differentiating.

12.6 DIRECTIONAL DERIVATIVES

In Section 12.3, we discussed the fact that if $f(x, y)$ is the temperature of a flat metal plate at the point $P(x, y)$ in an xy-plane, then the partial derivatives $f_x(x, y)$ and $f_y(x, y)$ give us the instantaneous rates of change of temperature with respect to distance in the horizontal and vertical directions, respectively (see Figures 12.28 and 12.29). In this section, we generalize this fact to the rate of change of $f(x, y)$ in *any* direction.

(b) A normal to edge AB is $\mathbf{j}$. In this case, AB is insulated if and only if

$$\nabla T \cdot \mathbf{j} = 0 \quad \text{or, equivalently,} \quad \frac{\partial T}{\partial y} = 0;$$

that is, the rate of change of T in the vertical direction is 0.

EXERCISES 12.6

Exer. 1–6: Find the gradient of f at P.

1 $f(x, y) = \sqrt{x^2 + y^2}$; $P(-4, 3)$

2 $f(x, y) = 7y - 5x$; $P(2, 6)$

3 $f(x, y) = e^{3x} \tan y$; $P(0, \pi/4)$

4 $f(x, y) = x \ln(x - y)$; $P(5, 4)$

5 $f(x, y, z) = yz^3 - 2x^2$; $P(2, -3, 1)$

6 $f(x, y, z) = xy^2e^z$; $P(2, -1, 0)$

c **Exer. 7–10: Use (12.24) or (12.30) to estimate the directional derivative of f at P in the indicated direction with $s = 0.02$, 0.01, and 0.005.**

7 $f(x, y) = \dfrac{x^3 \tanh(x + y)}{4 + x^2 + y^2}$; $P(2, 5)$, $\mathbf{a} = \sqrt{2}\mathbf{i} + \sqrt{3}\mathbf{j}$

8 $f(x, y) = x \ln(5x^2 + 4xy + y^2)$; $P(\sqrt{5}, 3)$, $\mathbf{a} = -0.89\mathbf{i} + 1.75\mathbf{j}$

9 $f(x, y, z) = y^2e^{(z^3+5xy)} + 6x^2yz$; $P(0, 1.2, -2.5)$, $\mathbf{a} = 3.7\mathbf{i} + 1.9\mathbf{j} - 2.1\mathbf{k}$

10 $f(x, y, z) = \dfrac{x^3 \sinh(xyz)}{1 + y^2 + z^4}$; $P(-1, 2, 1)$, $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$

Exer. 11–24: Find the directional derivative of f at the point P in the indicated direction.

11 $f(x, y) = x^2 - 5xy + 3y^2$; $P(3, -1)$, $\mathbf{u} = (\sqrt{2}/2)(\mathbf{i} + \mathbf{j})$

12 $f(x, y) = x^3 - 3x^2y - y^3$; $P(1, -2)$, $\mathbf{u} = \frac{1}{2}(-\mathbf{i} + \sqrt{3}\mathbf{j})$

13 $f(x, y) = \arctan \dfrac{y}{x}$; $P(4, -4)$, $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j}$

14 $f(x, y) = x^2 \ln y$; $P(5, 1)$, $\mathbf{a} = -\mathbf{i} + 4\mathbf{j}$

15 $f(x, y) = \sqrt{9x^2 - 4y^2 - 1}$; $P(3, -2)$, $\mathbf{a} = \mathbf{i} + 5\mathbf{j}$

16 $f(x, y) = \dfrac{x - y}{x + y}$; $P(2, -1)$, $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$

17 $f(x, y) = x \cos^2 y$; $P(2, \pi/4)$, $\mathbf{a} = \langle 5, 1 \rangle$

18 $f(x, y) = xe^{3y}$; $P(4, 0)$, $\mathbf{a} = \langle -1, 3 \rangle$

19 $f(x, y, z) = xy^3z^2$; $P(2, -1, 4)$, $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$

20 $f(x, y, z) = x^2 + 3yz + 4xy$; $P(1, 0, -5)$, $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

21 $f(x, y, z) = z^2e^{xy}$; $P(-1, 2, 3)$, $\mathbf{a} = 3\mathbf{i} + \mathbf{j} - 5\mathbf{k}$

22 $f(x, y, z) = \sqrt{xy} \sin z$; $P(4, 9, \pi/4)$, $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$

23 $f(x, y, z) = (x + y)(y + z)$; $P(5, 7, 1)$, $\mathbf{a} = \langle -3, 0, 1 \rangle$

24 $f(x, y, z) = z^2 \tan^{-1}(x + y)$; $P(0, 0, 4)$, $\mathbf{a} = \langle 6, 0, 1 \rangle$

Exer. 25–28: (a) Find the directional derivative of f at P in the direction from P to Q. (b) Find a unit vector in the direction in which f increases most rapidly at P, and find the rate of change of f in that direction. (c) Find a unit vector in the direction in which f decreases most rapidly at P, and find the rate of change of f in that direction.

25 $f(x, y) = x^2e^{-2y}$; $P(2, 0)$, $Q(-3, 1)$

26 $f(x, y) = \sin(2x - y)$; $P(-\pi/3, \pi/6)$, $Q(0, 0)$

27 $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$; $P(-2, 3, 1)$, $Q(0, -5, 4)$

28 $f(x, y, z) = \frac{x}{y} - \frac{y}{z}$; $P(0, -1, 2)$, $Q(3, 1, -4)$

29 A metal plate is located in an xy-plane such that the temperature T at (x, y) is inversely proportional to the distance from the origin, and the temperature at $P(3, 4)$ is 100 °F.

(a) Find the rate of change of T at P in the direction of $\mathbf{i} + \mathbf{j}$.

(b) In what direction does T increase most rapidly at P?

(c) In what direction does T decrease most rapidly at P?

(d) In what direction is the rate of change 0?

30 The surface of a lake is represented by a region D in the xy-plane such that the depth (in feet) under the point (x, y) is $f(x, y) = 300 - 2x^2 - 3y^2$.

(a) In what direction should a boat at $P(4, 9)$ sail in order for the depth of the water to decrease most rapidly?

(b) In what direction does the depth remain the same?

31 The electrical potential V at (x, y, z) is

$$V = x^2 + 4y^2 + 9z^2.$$

(a) Find the rate of change of V at $P(2, -1, 3)$ in the direction from P to the origin.

(b) Find the direction that produces the maximum rate of change of V at P.

(c) What is the maximum rate of change at P?

32 The temperature T at (x, y, z) is given by

$$T = 4x^2 - y^2 + 16z^2.$$

(a) Find the rate of change of T at $P(4, -2, 1)$ in the direction of $2\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$.

(b) In what direction does T increase most rapidly at P?

(c) What is this maximum rate of change?

(d) In what direction does T decrease most rapidly at P?

(e) What is this rate of change?

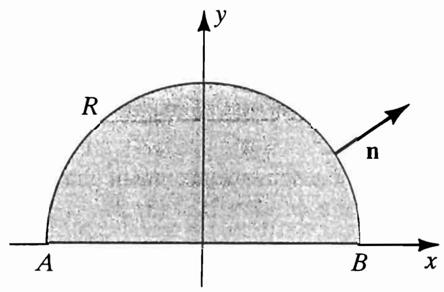
Exer. 33–34: Refer to the discussion that precedes Example 6. In each case, T is the temperature at (x, y).

33 Shown in the figure is a semicircular region R.

(a) Use polar coordinates to show that the upper boundary $\overset{\frown}{AB}$ is insulated if and only if $\partial T/\partial r = 0$. (*Hint:* Show that if $T = f(x, y)$, with $x = r\cos\theta$ and $y = r\sin\theta$, then $\partial T/\partial r = (\partial T/\partial x)\cos\theta + (\partial T/\partial y)\sin\theta$.)

(b) Interpret $\partial T/\partial r$ as a rate of change of T.

Exercise 33

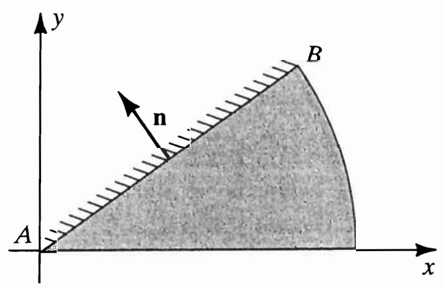


34 Shown in the figure is a circular sector whose boundary AB is insulated.

(a) Use polar coordinates to show that the insulation condition is equivalent to $\partial T/\partial\theta = 0$ for every point on the segment AB.

(b) Interpret $\partial T/\partial\theta$ as a rate of change of T.

Exercise 34



c 35 In some applications, it may be difficult to directly calculate the gradient $\nabla f(x, y)$. The following approximations are sometimes used for the components, where $h \approx 0$:

$$f_x(x, y) \approx \frac{f(x+h, y) - f(x-h, y)}{2h}$$

$$f_y(x, y) \approx \frac{f(x, y+h) - f(x, y-h)}{2h}$$

(a) Show that these approximations improve as $h \to 0$.

(b) If $f(x, y) = x^3/(1+y)$, approximate $\nabla f(1, 2)$ using $h = 0.01$ and compare the approximation with the exact result.

36 An analysis of the temperature of each component is crucial to the design of a computer chip. Suppose that for a chip to operate properly, the temperature of each component must not exceed 78 °F. If a component is likely to become too warm, engineers will usually place it in a cool portion of the chip. The designing of chips is aided by computer simulation in which temperature gradients are analyzed. A computer simulation for a new

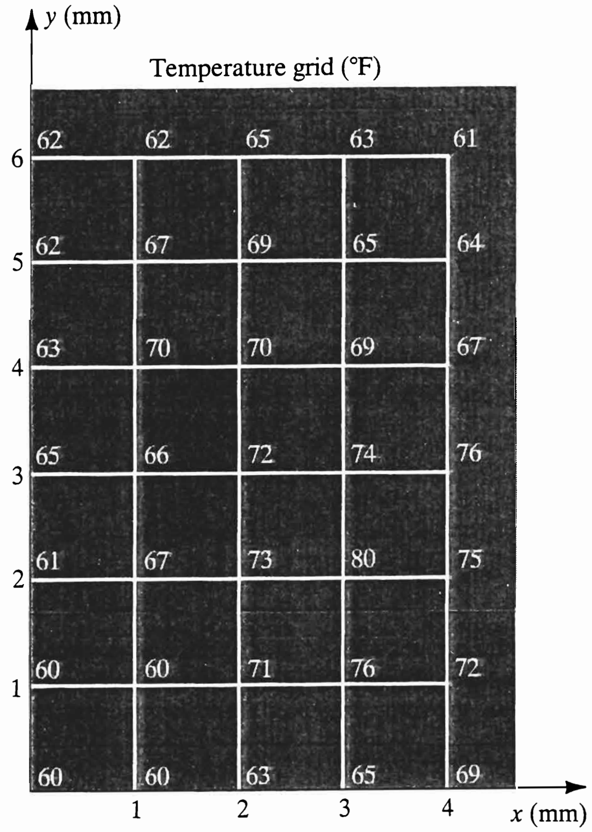
chip has resulted in the temperature grid (in °F) shown in the figure.

(a) If $T(x, y)$ is the temperature at (x, y), use Exercise 35 with $h = 1$ to approximate $\nabla T(3, 3)$.

(b) Estimate the direction of maximum heat transfer at $(3, 3)$.

(c) Estimate the instantaneous rate of change of T in the direction of $\mathbf{a} = -\mathbf{i} + 2\mathbf{j}$ at $(3, 3)$.

Exercise 36



c **Exer. 37–38: Extend the approximation formulas in Exercise 35 to the first partial derivatives of $f(x, y, z)$, and then use $h = 0.01$ to approximate the directional derivative of f at $P_0(1, 1, 1)$ in the direction of u.**

37 $f(x, y, z) = \dfrac{x^2 \sin y \tan z}{x^3 + y^2 z^4}$; $\quad \mathbf{u} = \dfrac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k})$

38 $f(x, y, z) = \dfrac{x^4 + 3x^2 z^2}{4x^2 y^2 + \cosh(yz)}$; $\quad \mathbf{u} = \dfrac{1}{\sqrt{6}}(2\mathbf{i} - \mathbf{j} + \mathbf{k})$

Exer. 39–44: If $u = f(x, y)$, $v = g(x, y)$, and f and g are differentiable, prove the identity.

39 $\nabla(cu) = c\,\nabla u$ for a constant c

40 $\nabla(u + v) = \nabla u + \nabla v$

41 $\nabla(uv) = u\,\nabla v + v\,\nabla u$

42 $\nabla\left(\dfrac{u}{v}\right) = \dfrac{v\,\nabla u - u\,\nabla v}{v^2}$ with $v \neq 0$

43 $\nabla u^n = nu^{n-1}\,\nabla u$ for every real number n

44 If $w = h(u)$, then $\nabla w = \dfrac{dw}{du}\,\nabla u$.

45 Let $\mathbf{u}$ be a unit vector and let θ be the angle, measured in the counterclockwise direction, from the positive x-axis to the position vector corresponding to $\mathbf{u}$.

(a) Show that

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta.$$

(b) If $f(x, y) = x^2 + 2xy - y^2$ and $\theta = 5\pi/6$, find $D_{\mathbf{u}} f(2, -3)$.

46 Refer to Exercise 45. If $f(x, y) = (xy + y^2)^4$ and $\theta = \pi/3$, find $D_{\mathbf{u}}(f(2, -1))$.

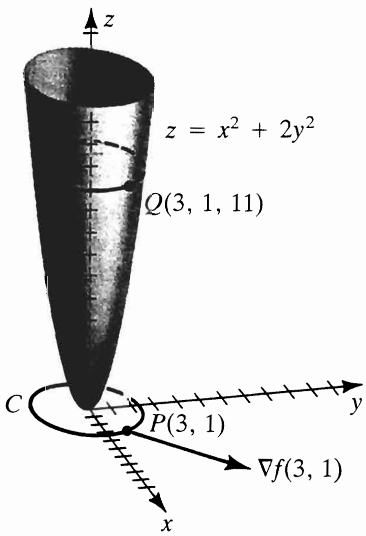
47 Suppose that $w = f(x, y)$, $x = g(t)$, $y = h(t)$, and all functions are differentiable. If $\mathbf{r}(t) = x\mathbf{i} + y\mathbf{j}$, prove that

$$\frac{dw}{dt} = \nabla w \cdot \mathbf{r}'(t).$$

12.7 TANGENT PLANES AND NORMAL LINES

In this section, we use the gradient of a function to study tangent planes to surfaces in three-dimensional space and their associated normal lines. Suppose that a surface S is the graph of an equation $F(x, y, z) = 0$ and that F has continuous first partial derivatives. Let $P_0(x_0, y_0, z_0)$ be a point on S at which F_x, F_y, and F_z are not all zero. A **tangent line** to S at P_0 is,

Figure 12.68



(b) The graph of f (that is, of $z = x^2 + 2y^2$) is an elliptic paraboloid (see Figure 12.68). The level curve $x^2 + 2y^2 = 11$ in the xy-plane corresponds to the trace of the paraboloid on the plane $z = 11$. The maximum rate of change of $f(x, y)$ occurs if the point (x, y) in the xy-plane moves in the direction of $\nabla f(3, 1)$ at $P(3, 1)$. This movement corresponds to movement of the point $(x, y, f(x, y))$ up the steepest part of the paraboloid at $Q(3, 1, 11)$.

Tangent planes can be used to obtain successive approximations to a solution of a system of two nonlinear equations, $f(x, y) = 0$, $g(x, y) = 0$, from a first approximation (x_1, y_1) by performing the following steps.

Step 1 Use Theorem (12.35) to find equations of the tangent planes to the graphs of f and g at the points $(x_1, y_1, f(x_1, y_1))$ and $(x_1, y_1, g(x_1, y_1))$.

Step 2 Find the trace in the xy-plane of each tangent plane in step (1).

Step 3 Find the point of intersection (x_2, y_2) of the traces found in step (2).

Step 4 Take (x_2, y_2) as a second approximation and repeat steps (1)–(3).

This process is a geometric description of Newton's method, discussed at the end of Section 12.4.

EXERCISES 12.7

Exer. 1–10: Find equations for the tangent plane and the normal line to the graph of the equation at the point *P*.

1 $4x^2 - y^2 + 3z^2 = 10$; $P(2, -3, 1)$

2 $9x^2 - 4y^2 - 25z^2 = 40$; $P(4, 1, -2)$

3 $z = 4x^2 + 9y^2$; $P(-2, -1, 25)$

4 $z = 4x^2 - y^2$; $P(5, -8, 36)$

5 $xy + 2yz - xz^2 + 10 = 0$; $P(-5, 5, 1)$

6 $x^3 - 2xy + z^3 + 7y + 6 = 0$; $P(1, 4, -3)$

7 $z = 2e^{-x} \cos y$; $P(0, \pi/3, 1)$

8 $z = \ln xy$; $P(\frac{1}{2}, 2, 0)$

9 $x = \ln \dfrac{y}{2z}$; $P(0, 2, 1)$

10 $xyz - 4xz^3 + y^3 = 10$; $P(-1, 2, 1)$

Exer. 11–14: Sketch both the level curve *C* of *f* that contains *P* and $\nabla f]_P$.

11 $f(x, y) = y^2 - x^2$; $P(2, 1)$

12 $f(x, y) = 3x - 2y$; $P(-2, 1)$

13 $f(x, y) = x^2 - y$; $P(-3, 5)$

14 $f(x, y) = xy$; $P(3, 2)$

Exer. 15–20: Sketch both the level surface *S* of *F* that contains *P* and $\nabla F]_P$.

15 $F(x, y, z) = x^2 + y^2 + z^2$; $P(1, 5, 2)$

16 $F(x, y, z) = z - x^2 - y^2$; $P(2, -2, 1)$

17 $F(x, y, z) = x + 2y + 3z$; $P(3, 4, 1)$

18 $F(x, y, z) = x^2 + y^2 - z^2$; $P(3, -1, 1)$

19 $F(x, y, z) = x^2 + y^2$; $P(2, 0, 3)$

20 $F(x, y, z) = z$; $P(2, 3, 4)$

Exer. 21–24: Prove that an equation of the tangent plane to the given quadric surface at the point $P_0(x_0, y_0, z_0)$ may be written in the indicated form.

21 $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$; $\dfrac{xx_0}{a^2} + \dfrac{yy_0}{b^2} + \dfrac{zz_0}{c^2} = 1$

22 $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$; $\dfrac{xx_0}{a^2} - \dfrac{yy_0}{b^2} + \dfrac{zz_0}{c^2} = 1$

23 $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1;$ $\dfrac{xx_0}{a^2} - \dfrac{yy_0}{b^2} - \dfrac{zz_0}{c^2} = 1$

24 $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = cz;$ $\dfrac{2xx_0}{a^2} + \dfrac{2yy_0}{b^2} = c(z + z_0)$

25 Find the points on the hyperboloid of two sheets with equation $x^2 - 2y^2 - 4z^2 = 16$ at which the tangent plane is parallel to the plane $4x - 2y + 4z = 5$.

26 Show that the sum of the squares of the x-, y-, and z-intercepts of every tangent plane to the graph of the equation $x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$ is the constant a^2.

27 Prove that every normal line to a sphere passes through the center of the sphere.

28 Find the points on the paraboloid $z = 4x^2 + 9y^2$ at which the normal line is parallel to the line through $P(-2, 4, 3)$ and $Q(5, -1, 2)$.

29 Two surfaces are said to be orthogonal at a point of intersection $P(x, y, z)$ if their normal lines at P are orthogonal. Show that the graphs of $F(x, y, z) = 0$ and $G(x, y, z) = 0$ (where F and G have partial derivatives) are orthogonal at P if and only if

$$F_xG_x + F_yG_y + F_zG_z = 0.$$

30 Refer to Exercise 29. Prove that the sphere with equation $x^2 + y^2 + z^2 = a^2$ and the cone $x^2 + y^2 - z^2 = 0$ are orthogonal at every point of intersection.

c **Exer. 31–34: For the given system of equations and first approximate solution (x_1, y_1), use Newton's method as outlined at the end of the section to find two more approximations.**

31 $x^2 - y^3 = 0,$ $x^2 + y^2 - 3 = 0;$ $(1.3, 1.1)$

32 $\sin x - \cos y = 0,$ $x^2 - y^2 - 1.3 = 0;$ $(1.2, 0.3)$

33 $2e^{(-x^2-y^2)} - 1 = 0,$ $4xy - x^4 - y^4 = 0;$ $(0.8, 0.2)$

34 $2\sin x \sin y - 1 = 0,$ $y^2 - x^2 + 1 = 0;$ $(1.1, 0.65)$

12.8 EXTREMA OF FUNCTIONS OF SEVERAL VARIABLES

In Chapter 3, we discussed local and absolute extrema for functions of one variable. In this section, we extend these concepts to functions of several variables.

A function f of two variables has a **local maximum** at (a, b) if there is an open disk R containing (a, b) such that $f(x, y) \le f(a, b)$ for every (x, y) in R. The local maxima correspond to the high points on the graph S of f, as illustrated in Figure 12.69. Similarly, the function f has a **local minimum** at (c, d) if there is an open disk R containing (c, d) such that $f(x, y) \ge f(c, d)$ for every (x, y) in R. The local minima correspond to the low points on the graph of f, as illustrated in Figure 12.70.

Figure 12.69 Local maximum $f(a, b)$

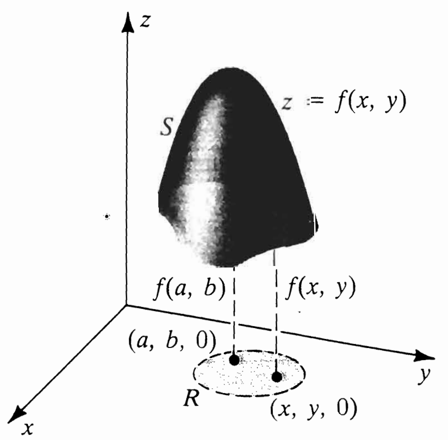
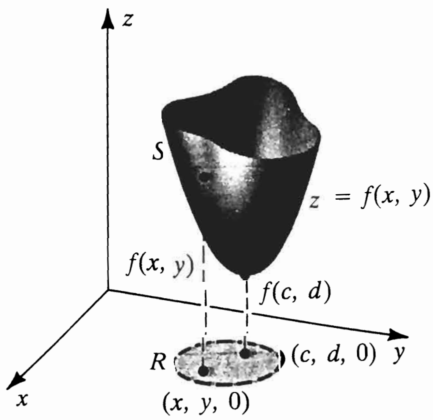


Figure 12.70 Local minimum $f(c, d)$



Let θ be the angle between the vectors

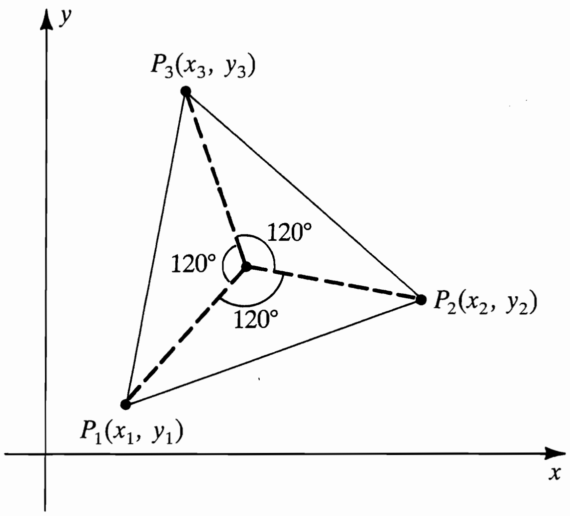
$$\overrightarrow{P_1P} = (x - x_1)\mathbf{i} + (y - y_1)\mathbf{j} \quad \text{and} \quad \overrightarrow{P_2P} = (x - x_2)\mathbf{i} + (y - y_2)\mathbf{j}.$$

Applying Corollary (10.20) and using (*), we have

$$\cos\theta = \frac{\overrightarrow{P_1P} \cdot \overrightarrow{P_2P}}{\|\overrightarrow{P_1P}\| \|\overrightarrow{P_2P}\|} = -\frac{1}{2}.$$

Hence, $\theta = \arccos(-\frac{1}{2}) = 120°$. By symmetry, the angles between $\overrightarrow{P_1P}$ and $\overrightarrow{P_3P}$ and between $\overrightarrow{P_2P}$ and $\overrightarrow{P_3P}$ must also equal 120°, as illustrated in Figure 12.76.

Figure 12.76



The discussion in this section can be generalized to functions of more than two variables. For example, given $f(x, y, z)$, we define local maxima and minima in a manner analogous to that used for the two-variable case. If f has first partial derivatives, then a local extremum can occur only at a point where f_x, f_y, and f_z are simultaneously 0. It is difficult to obtain tests for determining whether such a point corresponds to a maximum, a minimum, or neither. However, in applications we can often determine this by analyzing the physical nature of the problem.

EXERCISES 12.8

Exer. 1–20: Find the extrema and saddle points of f.

1 $f(x, y) = -x^2 - 4x - y^2 + 2y - 1$

2 $f(x, y) = x^2 - 2x + y^2 - 6y + 12$

3 $f(x, y) = x^2 + 4y^2 - x + 2y$

4 $f(x, y) = 5 + 4x - 2x^2 + 3y - y^2$

5 $f(x, y) = x^2 + 2xy + 3y^2$

6 $f(x, y) = x^2 - 3xy - y^2 + 2y - 6x$

7 $f(x, y) = x^3 + 3xy - y^3$

8 $f(x, y) = x^2 + xy$

9 $f(x, y) = \frac{1}{2}x^2 + 2xy - \frac{1}{2}y^2 + x - 8y$

10 $f(x, y) = -2x^2 - 2xy - \frac{3}{2}y^2 - 14x - 5y$

11 $f(x, y) = \frac{1}{3}x^3 - \frac{2}{3}y^3 + \frac{1}{2}x^2 - 6x + 32y + 4$

12 $f(x, y) = \frac{1}{3}x^3 + \frac{1}{3}y^3 - \frac{3}{2}x^2 - 4y$

13 $f(x, y) = \frac{1}{2}x^4 - 2x^3 + 4xy + y^2$

14 $f(x, y) = \frac{1}{3}x^3 + 4xy - 9x - y^2$

15 $f(x, y) = x^4 + y^3 + 32x - 9y$

16 $f(x, y) = -\frac{1}{3}x^3 + xy + \frac{1}{2}y^2 - 12y$

17 $f(x, y) = e^x \sin y$

18 $f(x, y) = x \sin y$

19 $f(x, y) = \dfrac{4y + x^2y^2 + 8x}{xy}$

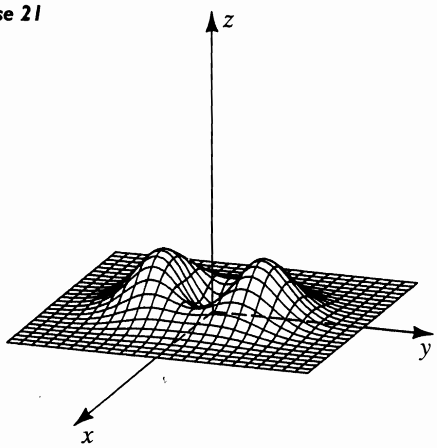
20 $f(x, y) = \dfrac{x}{x + y}$

21 Shown in the figure is a graph of

$$f(x, y) = (x^2 + 3y^2)e^{-(x^2+y^2)}.$$

Show that there are five critical points, and find the extrema of f.

Exercise 21



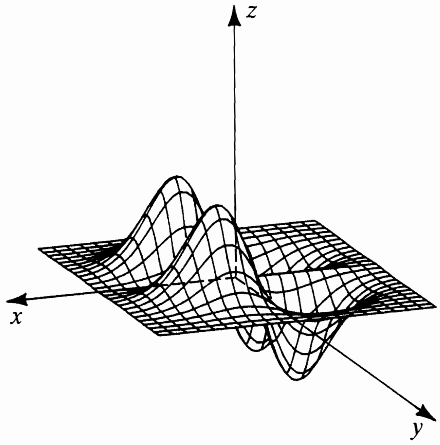
22 Shown in the figure is a graph of

$$f(x, y) = xy^2e^{-(x^2+y^2)/4}.$$

(a) Show that there is an infinite number of critical points.

(b) Find the coordinates of the four critical points shown in the figure.

Exercise 22



Exer. 23–28: Find the maximum and minimum values of f on R. (Refer to Exercises 3–8 for local extrema.)

23 $f(x, y) = x^2 + 4y^2 - x + 2y$;
the region R bounded by the ellipse $x^2 + 4y^2 = 1$

24 $f(x, y) = 5 + 4x - 2x^2 + 3y - y^2$;
the triangular region R bounded by the lines $y = x$, $y = -x$, and $y = 2$

25 $f(x, y) = x^2 + 2xy + 3y^2$;
$R = \{(x, y): -2 \le x \le 4, -1 \le y \le 3\}$

26 $f(x, y) = x^2 - 3xy - y^2 + 2y - 6x$;
$R = \{(x, y): |x| \le 3, |y| \le 2\}$

27 $f(x, y) = x^3 + 3xy - y^3$;
the triangular region R with vertices $(1, 2)$, $(1, -2)$, and $(-1, -2)$

28 $f(x, y) = x^2 + xy$;
the region R bounded by the graphs of $y = x^2$ and $y = 9$

29 Find the shortest distance from the point $P(2, 1, -1)$ to the plane $4x - 3y + z = 5$.

30 Find the shortest distance between the parallel planes $2x + 3y - z = 2$ and $2x + 3y - z = 4$.

31 Find the points on the graph of $xy^3z^2 = 16$ that are closest to the origin.

32 Find three positive real numbers whose sum is 1000 and whose product is a maximum.

33 If an open rectangular box is to have a fixed volume V, what relative dimensions will make the surface area a minimum?

34 If an open rectangular box is to have a fixed surface area A, what relative dimensions will make the volume a maximum?

35 Find the dimensions of the rectangular box of maximum volume with faces parallel to the coordinate planes that can be inscribed in the ellipsoid

$$16x^2 + 4y^2 + 9z^2 = 144.$$

36 Generalize Exercise 35 to any ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

37 Find the dimensions of the rectangular box of maximum volume that has three of its faces in the coordinate planes, one vertex at the origin, and another vertex in the first octant on the plane $4x + 3y + z = 12$.

38 Generalize Exercise 37 to any plane

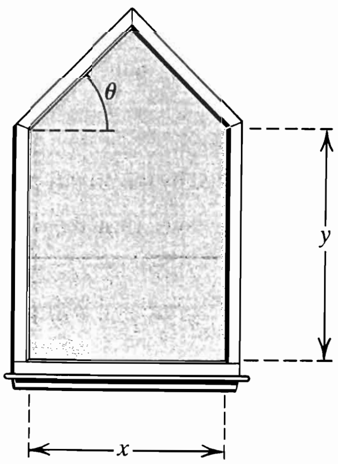
$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1,$$

where a, b, and c are positive real numbers.

39 A company plans to manufacture closed rectangular boxes that have a volume of 8 ft^3. Find the dimensions that will minimize the cost if the material for the top and bottom costs twice as much as the material for the sides.

40 A window has the shape of a rectangle surmounted by an isosceles triangle, as illustrated in the figure. If the perimeter of the window is 12 ft, what values of x, y, and θ will maximize the total area?

Exercise 40



41 The U.S. Postal Service will not accept a rectangular box if the sum of its length and girth (the perimeter of a cross section that is perpendicular to the length) is more than 108 in. Find the dimensions of the box of maximum volume that can be mailed.

42 Find a vector in three dimensions having magnitude 8 such that the sum of its components is as large as possible.

Exer. 43 – 44: Refer to Example 7.

43 Three electrical components of a computer are located at $P_1(0, 0)$, $P_2(4, 0)$, and $P_3(0, 4)$. Locate the position of a fourth component so that the signal delay time is minimal.

44 Show that if triangle $P_1P_2P_3$ contains an angle that is greater than or equal to 120°, then P cannot be located as in Figure 12.76.

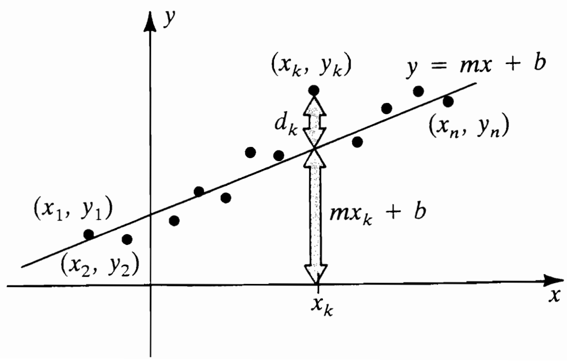
45 In scientific experiments, corresponding values of two quantities x and y are often tabulated as follows:

x-values	x_1	x_2	$\cdots$	x_n
y-values	y_1	y_2	$\cdots$	y_n

Plotting the points (x_k, y_k) may lead the investigator to conjecture that x and y are related *linearly*—that is, $y = mx + b$ for some m and b. Thus, it is desirable to find a line l having that equation which best fits the data, as illustrated in the figure. Statisticians call l a *linear regression line.*

One technique for finding l is to use the *method of least squares.* To do so, consider, for each k, the *vertical deviation* $d_k = y_k - (mx_k + b)$ of the point (x_k, y_k) from the line $y = mx + b$ (see figure). Values of m and b are then determined that minimize the sum of the squares $\sum_{k=1}^{n} d_k^2$ (the squares d_k^2 are used because some

Exercise 45



of the d_k may be negative). Substituting for d_k produces the following function f of m and b:

$$f(m, b) = \sum_{k=1}^{n} (y_k - mx_k - b)^2$$

Show that the line $y = mx + b$ of best fit occurs if

$$\left(\sum_{k=1}^{n} x_k\right) m + nb = \sum_{k=1}^{n} y_k$$

and

$$\left(\sum_{k=1}^{n} x_k^2\right) m + \left(\sum_{k=1}^{n} x_k\right) b = \sum_{k=1}^{n} x_k y_k.$$

Thus, the line can be found by solving this system of two equations for the two unknowns m and b.

46 Given the equations in Exercise 45, show that the sum $\sum_{k=1}^{n} d_k$ of the deviations is 0. (This means that the positive and negative deviations cancel one another, and it is one reason for using $\sum_{k=1}^{n} d_k^2$ in the method of least squares.)

Exer. 47–48: Use the method of least squares (see Exercise 45) to find a line $y = mx + b$ that best fits the given data.

47

x-values	1	4	7
y-values	3	5	6

48

x-values	1	4	6	8
y-values	1	3	2	4

c **49** The following table lists the relationship between semester averages and scores on the final examination for ten students in a mathematics class.

Semester average	40	55	62	68	72	76	80	86	90	94
Final examination	30	45	65	72	60	82	76	92	88	98

Fit these data to a line, and use the line to estimate the final examination grade of a student with an average of 70.

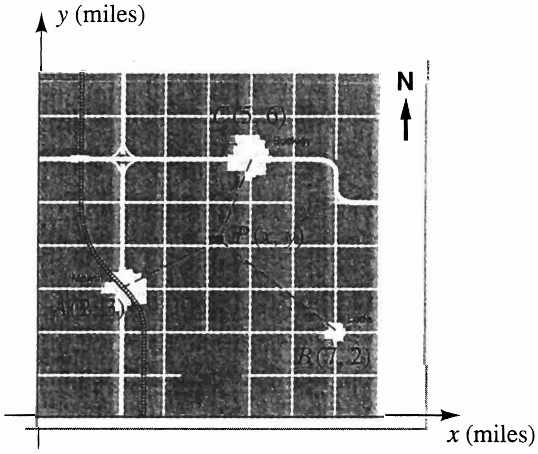
c **50** In studying the stress–strain diagram of an elastic material, an engineer finds that part of the curve appears to be linear. Experimental values are listed in the following table.

Stress (lb)	2	2.2	2.4	2.6	2.8	3.0
Strain (in.)	0.10	0.30	0.40	0.60	0.70	0.90

Fit these data to a straight line, and estimate the strain when the stress is 2.5 lb.

51 Shown in the figure are the relative positions of three towns, A, B, and C. City planners want to use the least-squares criterion to decide where to construct a new high school that will serve all three communities. They will construct the school about a point $P(x, y)$ at which the sum of the squares of the distances from towns A, B, and C is a minimum. Find the relative position of the construction site.

Exercise 51



52 Generalize Exercise 51 to the case of n towns at positions $Q_1(x_1, y_1), Q_2(x_2, y_2), \ldots, Q_n(x_n, y_n)$.

53 Exercise 45 may be generalized to solve the problem of finding the plane $z = ax + by + c$ that best fits data $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots, (x_n, y_n, z_n)$. The method of least squares attempts to determine values a, b, and c such that

$$f(a, b, c) = \sum_{k=1}^{n} (z_k - ax_k - by_k - c)^2$$

is minimized.

(a) Find a system of three equations for the three unknowns a, b, and c.

(b) Find the plane that best fits the data (0, 0, 0), (0, 1, 0), (0, 0, 1), and (1, 1, 2). (*Hint:* Solve the system of three equations $f_a = 0$, $f_b = 0$, $f_c = 0$.)

54 Three alleles (alternative forms of a gene) A, B, and O determine the four human blood types: A (AA or AO), B (BB or BO), O (OO), and AB. The *Hardy–Weinberg law* asserts that the proportion of individuals in a population who carry two different alleles is given by the formula

$$P = 2pq + 2pr + 2rq,$$

where p, q, and r are the proportions of alleles A, B, and O, respectively, in the population. Show that P must be less than or equal to $\frac{2}{3}$. (*Hint:* $p \geq 0, q \geq 0, r \geq 0$, and $p + q + r = 1$.)

c **Exer. 55–56: Estimate the critical points of f on $R = \{(x, y): |x| \leq 1.5 \text{ and } |y| \leq 1.5\}$ by graphing $f_x(x, y) = 0$ and $f_y(x, y) = 0$ on the same coordinate plane.**

55 $f(x, y) = x^3 \sin x - xy + 4y^2 + y$

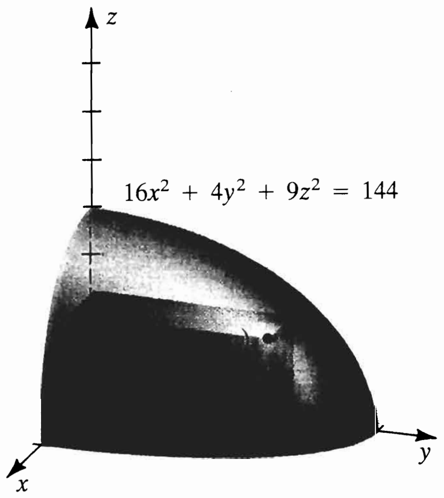
56 $f(x, y) = xy - \arctan x - y^{5/4}$

12.9 LAGRANGE MULTIPLIERS

In many applications, we must find the extrema of a function f of several variables when the variables are restricted in some manner. In this section, we introduce the technique of *Lagrange multipliers** to locate such extrema. As an illustration, suppose we wish to find the volume of the largest rectangular box with faces parallel to the coordinate planes that can be inscribed in the ellipsoid $16x^2 + 4y^2 + 9z^2 = 144$. Note that, by symmetry, it is sufficient to examine the part in the first octant illustrated in Figure 12.77. If $P(x, y, z)$ is the vertex shown in the figure, then the volume V of the entire box is $V = 8xyz$. We must find the maximum value of V subject to the **constraint** (or **side condition**)

$$16x^2 + 4y^2 + 9z^2 - 144 = 0.$$

Figure 12.77



*This method was invented by the French mathematician Joseph-Louis Lagrange (see *Mathematicians and Their Times*, Chapter 7).

Subtracting the second equation from the first, we obtain the following equivalent equations:

$$2x - 2y = (2x\lambda + \mu) - (2y\lambda + \mu) = 2x\lambda - 2y\lambda$$
$$2(x - y) - 2\lambda(x - y) = 0$$
$$2(x - y)(1 - \lambda) = 0$$

Consequently, either $\lambda = 1$ or $x = y$.

If $\lambda = 1$, we have $2z = 2\lambda = 2(1)$, or $z = 1$. The first constraint—that is, $x^2 + y^2 + 2z - 16 = 0$—then gives us $x^2 + y^2 - 14 = 0$. Solving this equation simultaneously with $x + y - 4 = 0$, we find that either

$$x = 2 + \sqrt{3}, \quad y = 2 - \sqrt{3} \quad \text{or} \quad x = 2 - \sqrt{3}, \quad y = 2 + \sqrt{3}.$$

Thus, points on C that may lead to extrema are

$$P_1(2 + \sqrt{3}, 2 - \sqrt{3}, 1) \quad \text{and} \quad P_2(2 - \sqrt{3}, 2 + \sqrt{3}, 1).$$

The corresponding distances from O are

$$d(O, P_1) = \sqrt{15} = d(O, P_2).$$

If $x = y$, then, using the constraint $x + y - 4 = 0$, we obtain the equivalent equations $x + x - 4 = 0$, $2x = 4$, or $x = 2$. This gives us $P_3(2, 2, 4)$ and $d(O, P_3) = 2\sqrt{6}$.

Referring to Figure 12.81, we may now make the following observations. As a point moves continuously along C from $A(4, 0, 0)$ to $B(0, 4, 0)$, its distance from the origin starts at $d(O, A) = 4$, decreases to a minimum value $\sqrt{15}$ at P_1, and then increases to a maximum value $2\sqrt{6}$ at P_3. The distance then decreases to $\sqrt{15}$ at P_2 and again increases to 4 at B.

As a check on the solution, note that parametric equations for C are

$$x = 4 - t, \quad y = t, \quad z = 4t - t^2; \quad 0 \le t \le 4.$$

In this case,

$$f(x, y, z) = (4 - t)^2 + t^2 + (4t - t^2)^2,$$

and the extrema of f may be found using single-variable methods. It can be verified that the same points are obtained.

EXERCISES 12.9

Exer. 1 – 10: Use Lagrange multipliers to find the extrema of f subject to the stated constraints.

1 $f(x, y) = y^2 - 4xy + 4x^2$; $x^2 + y^2 = 1$

2 $f(x, y) = 2x^2 + xy - y^2 + y$; $2x + 3y = 1$

3 $f(x, y, z) = x + y + z$; $x^2 + y^2 + z^2 = 25$

4 $f(x, y, z) = x^2 + y^2 + z^2$; $x + y + z = 25$

5 $f(x, y, z) = x^2 + y^2 + z^2$; $x - y + z = 1$

6 $f(x, y, z) = x + 2y - 3z;$
$$z = 4x^2 + y^2$$

7 $f(x, y, z) = x^2 + y^2 + z^2;$
$$x - y = 1, \quad y^2 - z^2 = 1$$

8 $f(x, y, z) = z - x^2 - y^2;$
$$x + y + z = 1, \quad x^2 + y^2 = 4$$

9 $f(x, y, z, t) = xyzt;$
$$x - z = 2, \quad y^2 + t = 4$$

10 $f(x, y, z, t) = x^2 + y^2 + z^2 + t^2;$
$$3x + 4y = 5, \quad z + t = 2$$

11 Find the point on the sphere $x^2 + y^2 + z^2 = 9$ that is closest to the point (2, 3, 4).

12 Find the point on the line of intersection of the planes $x + 3y - 2z = 11$ and $2x - y + z = 3$ that is closest to the origin.

13 A closed rectangular box having a volume of 2 ft^3 is to be constructed. If the cost per square foot of the material for the sides, bottom, and top is \$1.00, \$2.00, and \$1.50, respectively, find the dimensions that will minimize the cost.

14 Prove that a closed rectangular box of fixed volume and minimal surface area is a cube.

15 Find the volume of the largest rectangular box that has three of its vertices on the positive x-, y-, and z-axes, respectively, and a fourth vertex on the plane $2x + 3y + 4z = 12$.

16 Find the dimensions of the rectangular box of maximum volume that has three of its faces in the coordinate planes, one vertex at the origin, and another vertex in the first octant on the plane $2x + 3y + 5z = 90$.

17 A container with a closed top and fixed surface area is to be constructed in the shape of a right circular cylinder. Find the relative dimensions that maximize the volume.

18 Find the dimensions of the rectangular box of maximum volume, with faces parallel to the coordinate planes, that can be inscribed in the ellipsoid $4x^2 + 4y^2 + z^2 = 36$.

19 Prove that the triangle of maximum area and fixed perimeter p is equilateral. (*Hint:* If the sides are x, y, z and if $s = \frac{1}{2}p$, then the area A is given by Heron's formula, $A = \sqrt{s(s-x)(s-y)(s-z)}$.)

20 Prove that the product of the sines of the angles of a triangle is greatest when the triangle is equilateral.

21 The strength of a rectangular beam varies as the product of its width and the square of its depth. Find the dimensions of the strongest rectangular beam that can be cut from a cylindrical log whose cross sections are elliptical with major and minor axes of lengths 24 in. and 16 in., respectively.

22 If x units of capital and y units of labor are required to manufacture $f(x, y)$ units of a certain commodity, the *Cobb–Douglas production function* is defined by $f(x, y) = kx^a y^b$, where k is a constant and a and b are positive numbers such that $a + b = 1$. Suppose that $f(x, y) = x^{1/5}y^{4/5}$ and that each unit of capital costs C dollars and each unit of labor costs L dollars. If the total amount available for these costs is M dollars, so $xC + yL = M$, how many units of capital and labor will maximize production?

c **Exer. 23–24: Use Lagrange multipliers and graphs to estimate the extrema of $f(x, y)$ subject to the constraint $g(x, y) = 0$.**

23 $f(x, y) = y - \cos x + 2x;$ $\quad g(x, y) = x^2 + 2y^2 - 1$

24 $f(x, y) = \frac{1}{5}x^5 + \frac{1}{3}y^3;$ $\quad g(x, y) = x^2 + y^2 - 1$

CHAPTER 12 REVIEW EXERCISES

Exer. 1–4: Describe the domain of f and the level curve or surface through P.

1 $f(x, y) = \sqrt{36 - 4x^2 + 9y^2};$ $\quad P(3, 4)$

2 $f(x, y) = \ln xy;$ $\quad P(2, 3)$

3 $f(x, y, z) = (z^2 - x^2 - y^2)^{-3/2};$ $\quad P(0, 0, 1)$

4 $f(x, y, z) = \dfrac{\sec z}{x - y};$ $\quad P(5, 3, 0)$