FINAL EXAMINATION, SEMESTER I, 1445

DEPT. MATH., COLLEGE OF SCIENCE, KSU

MATH: 107 FULL MARK: 40 TIME: 3 HOURS

Q1. [Marks: 3+4=7]

(a) Find conditions on a, b such that the following system is consistent:

$$x + 2y + z + 3t = a$$

$$2x + y + 3z + 2t = b$$

$$-x + 7y - 4z + 9t = 1.$$

(b) Evaluate the determinant by reducing the matrix to row echelon form:

$$\begin{bmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & -4 & 7 & 1 \end{bmatrix}$$

Q2. [Marks: 2+3+3+3=11]

(a) Let g be a function of two variables defined on $\Re^2$ by: $w = g(x, y) = \sin(x - y)$. Show that the function satisfies the Laplace equation: $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0$.

(b) Find symmetric equations for the line passing through point $P(4, 3, 2)$ and parallel to the vector $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

(c) Identify the surface $9(x^2 + z^2) + 4y^2 = 36$. Give its traces, and sketch.

(d) Let C be the curve with parametric equations $x = t$, $y = t^2$, $z = t^3$, $t \geq 0$. Find the parametric equations for the tangent line to C at the point corresponding to $t = 2$.

Q3. [Mark: 3+4+3=10]

(a) A moving particle in the space has position vector $\mathbf{r}(t) = \langle 2\sin t, \sin^2 t, t - \sin t \cos t \rangle$. Show that the speed of the particle is constant during the motion.

(b) A moving particle in the plane has position vector $\mathbf{r}(t) = \langle \ln(1 + t^2), 2\tan^{-1} t - t \rangle$. Find unit tangent vector $\mathbf{T}(t)$ and the principal unit normal vector $\mathbf{N}(t)$.

(c) Find the position vector $\mathbf{r}(t)$ of a moving particle with acceleration $\mathbf{a}(t) = \langle 6t + 2, e^t - 2, -\sin t \rangle$ and initial position vector and velocity by $\mathbf{r}(0) = \langle 1, -1, 0 \rangle$ and $\mathbf{v}(0) = \langle 1, 0, 0 \rangle$, respectively.

Q4. [Marks: 2+3+4+3=12]

(a) Let $z = x^2 y$ where $x = r\cos\theta$ and $y = r\sin\theta$. Use Chain rule to find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$.

(b) Let $f(x, y, z) = 4x - y^2 e^{3xz}$. Find the directional derivative of f at the point $A(3, -1, 0)$ in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

(c) If $g(x, y) = x^2 + y^2 + xy^2 + 16$. Find the local extrema and saddle point, if any.

(d) Let $f(x, y, z) = xyz$ where x,y and z are three positive numbers. Use Lagrange multiplier to maximize f subject to the constraint $x + y + z = 1$.

Solution M 107

Q1. a)

$$\begin{pmatrix} 1 & 2 & 1 & 3 & a \\ 2 & 1 & 3 & 2 & b \\ -1 & 7 & -4 & 9 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & 3 & a \\ 0 & -3 & 1 & -4 & b-2a \\ 0 & 0 & 0 & 0 & a+1+3b-6a \end{pmatrix}$$

(2)

The system is consistent if $a+1+3b-6a=0$ (1)

i.e. $3b-5a+1=0$ or $a=\frac{1}{5}(3b+1)$ (1)

b) (2)

$$\begin{vmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & -4 & 7 & 1 \end{vmatrix} \begin{matrix} -5r_1+r_2 \\ = \\ r_1+r_3 \\ -2r_1+r_4 \end{matrix} \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 1 & -1 \end{vmatrix} \; 3r_4+r_3$$

(1)

$$= \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & -4 \end{vmatrix} = 12$$

(1)

Q2. a) $w = \sin(x-y)$

$$\frac{\partial w}{\partial x} = \cos(x-y)$$

$$\Rightarrow \frac{\partial^2 w}{\partial x^2} = -\sin(x-y)$$

(1)

$$\frac{\partial w}{\partial y} = \cos(x-y)(-1)$$

(1)

$$\frac{\partial^2 w}{\partial y^2} = -\sin(x-y) \quad \Rightarrow \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} = 0$$

Q2 b) $\underline{a} = \langle 1, 2, 3 \rangle$

The parametric eqs are

$x = 4 + t, \; y = 3 + 2t, \; z = 2 + 3t$ (3)

so the symmetric form is

$$\frac{x-4}{1} = \frac{y-3}{2} = \frac{z-2}{3}.$$

c) Please see the solution at page 6

d) $C: \begin{cases} x = t \\ y = t^2 \\ z = t^3, \quad t \geq 0 \end{cases}$

$\underline{r}(t) = t\underline{i} + t^2\underline{j} + t^3\underline{k}$ (1)

$\Rightarrow \underline{r}'(t) = \underline{i} + 2t\underline{j} + 3t^2\underline{k}$ (1)

At $t=2$, the tangent vector is $\underline{r}'(2) = \langle 1, 4, 12 \rangle$

and the point corresponds to $t=2$ is $P(2,4,8)$.

So, the parametric eqs for the tangent line is given by $x = 2+t, \; y = 4 + 4t, \; z = 8 + 12t$ (1)

Q3 a) $\underline{r}'(t) = \langle 2\cos t, 2\cos t \sin t, 1 - \cos^2 t + \sin^2 t \rangle$

Therefore, $\|\underline{r}'(t)\| = \sqrt{4\cos^2 t + 4\cos^2 t \sin^2 t + 4\sin^4 t}$

$= 2\sqrt{\cos^2 t + \sin^2 t} = 2$

(2 + 1)

Q_3 b) $\underline{v}(t) = \left\langle \frac{2t}{1+t^2}, \frac{2}{1+t^2} - 1 \right\rangle$

$= \frac{1}{1+t^2}\left\langle 2t, 1-t^2 \right\rangle$ (2)

$\therefore \|\underline{v}(t)\| = \frac{1}{1+t^2}\sqrt{4t^2+1-2t^2+t^4}$

$= \frac{1}{1+t^2}\sqrt{1+2t^2+t^4} = 1$

$\underline{T}(t) = \frac{1}{\|\underline{v}(t)\|}\underline{v}(t) = \left\langle \frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2} \right\rangle$ (1)

$\underline{N}(t) = \left\langle \frac{1-t^2}{1+t^2}, \frac{-2t}{1+t^2} \right\rangle$ (1)

c) $\underline{v}(t) = \int \underline{a}(t)\,dt = \langle 3t^2+2t+c_1, e^t-2t+c_2, \cos t + c_3 \rangle$

and $\underline{v}(0) = \langle c_1, 1+c_2, 1+c_3 \rangle = \langle 1, 0, 0 \rangle$

$\Rightarrow \underline{v}(t) = \langle 3t^2+2t+1, e^t-2t-1, \cos t - 1 \rangle$ (2)

$\underline{r}(t) = \int \underline{v}(t)\,dt$

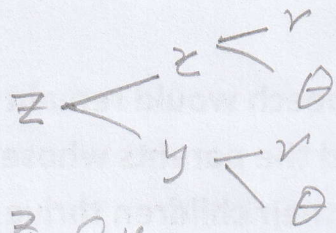
$= \langle t^3+t^2+t+c_1, e^t-t^2-t+c_2, \sin t - t + c_3 \rangle$

$\underline{r}(0) = \langle c_1, 1+c_2, c_3 \rangle = \langle 1, -1, 0 \rangle$

$\therefore \underline{r}(t) = \langle t^3+t^2+t+1, e^t-t^2-t-2, \sin t - t \rangle$ (1)

Q4 a) $x = r\cos\theta, \; y = r\sin\theta$

$z = x^2 y$



$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r}$$

1.5

$$= (2xy)(\cos\theta) + (x^2)(\sin\theta)$$

$$= 2(r\cos\theta)(r\sin\theta)\cos\theta + (r^2\cos^2\theta)(\sin\theta)$$

$$= 2r^2\cos^2\theta\sin\theta + r^2\cos^2\theta\sin\theta$$

$$= 3r^2\cos^2\theta\sin\theta$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial \theta}$$

$$= (2xy)(-r\sin\theta) + (x^2)(r\cos\theta)$$

$$= (2r^2\cos\theta\sin\theta)(-r\sin\theta) + (r^2\cos^2\theta)(r\cos\theta)$$

$$= -2r^3\cos\theta\sin^2\theta + r^3\cos^3\theta$$

1.5

b) $\nabla f = \langle 4 - 3zy^2 e^{3xz}, -2y e^{3xz}, -3xy^2 e^{3xz} \rangle$

1.5

$$\nabla f(3,-1,0) = \langle 4, 2, -9 \rangle$$

$$\underline{u} = \frac{\langle 2,2,1 \rangle}{\sqrt{2^2+2^2+1}} = \left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle$$

1.5

$$D_u f(3,-1,0) = \nabla f \cdot \underline{u} \quad \langle 4,2,-9 \rangle \left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle$$

$$= \langle 4, 2, -9 \rangle$$

$$= \frac{8}{3} + \frac{4}{3} - \frac{9}{3} = 1$$

(1)

Q_4 c) $g(x,y) = x^2 + y^2 + xy^2 + 9$

$$\left.\begin{aligned} g_x(x,y) &= 2x + y^2 = 0 \\ g_y(x,y) &= 2y + 2xy = 0 \end{aligned}\right\} \Rightarrow y=0,\ x=-1$$

If $y=0$, $x=0$

If $x=-1$, $y=\sqrt{2}$ or $y=-\sqrt{2}$

Critical points are: $(-1,\sqrt{2}), (-1,-\sqrt{2}), (0,0)$ (1)

$$D(x,y) = \begin{pmatrix} 2 & 2y \\ 2y & 2+2x \end{pmatrix} \quad (1)$$

(2)

$D(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 4$, $D(0,0) = 4 > 0$

$g_{xx}(0,0) = 2 > 0$

At $(0,0)$, local min.

At $(-1,\sqrt{2})$, $D(-1,\sqrt{2}) = -8 < 0$ saddle point

At $(-1,-\sqrt{2})$, $D(-1,-\sqrt{2}) = -8 < 0$ saddle point

d) $f(x,y,z) = xyz$

$g(x,y,z) = x+y+z = 1$

$\nabla f = \lambda \nabla g$

$\langle yz, xz, xy \rangle = \langle \lambda, \lambda, \lambda \rangle$ (2)

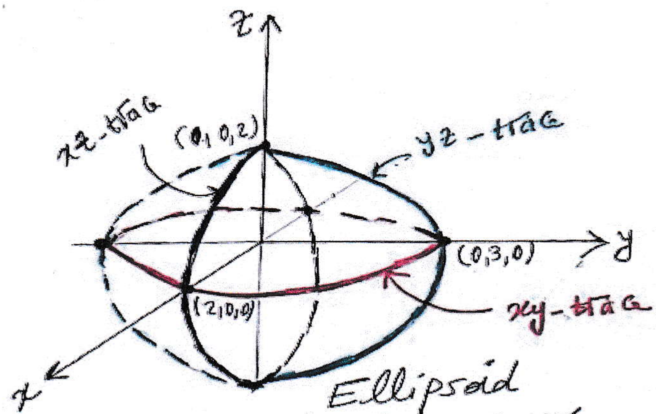
$$\left.\begin{aligned} yz &= \lambda \\ xz &= \lambda \\ xy &= \lambda \end{aligned}\right\} \Rightarrow x = y = z$$

Then $3x = 1 \Rightarrow x = y = z = \frac{1}{3}$

(1)

Q2 c)

Trace	**Equation of trace**	**Description**
On xy-plane ($z = 0$)	$\frac{x^2}{4} + \frac{y^2}{9} = 1$	Ellipse
On yz-plane ($x = 0$)	$\frac{y^2}{9} + \frac{z^2}{4} = 1$	Ellipse
On xz-plane ($y = 0$)	$\frac{x^2}{4} + \frac{z^2}{4} = 1$ $\Rightarrow x^2 + z^2 = 4$	Circle



Ellipsoid

$9(x^2+z^2)+4y^2=36$

$\Rightarrow \frac{x^2}{4}+\frac{z^2}{4}+\frac{y^2}{9}=1$