King Saud University
College of Science
Department of Mathematics

M 106 - INTEGRAL CALCULUS

Solutions of the first midterm exam
Second Semester 1432-1433 H

Dr. Tariq A. AlFadhel[1][2]

[1] E-mail : alfadhel@ksu.edu.sa
[2] URL : http://faculty.ksu.edu.sa/alfadhel

Multiple choice questions (One mark for each question)

Question 1. If $\sum_{k=1}^{25}(k^2 + \alpha k) = 0$, then the value of α is equal to :

(a) -16

(b) 16

(c) -17

(d) -1

Answer: $\sum_{k=1}^{25}(k^2 + \alpha k) = 0 \Rightarrow \left(\sum_{k=1}^{25} k^2\right) + \alpha \sum_{k=1}^{25} k = 0$

$$\Rightarrow \sum_{k=1}^{25} k^2 = -\alpha \sum_{k=1}^{25} k \Rightarrow \alpha = -\frac{\sum_{k=1}^{25} k^2}{\sum_{k=1}^{25} k} = -\frac{\frac{(25)(25+1)(2(25)+1)}{6}}{\frac{25(25+1)}{2}} = -\frac{51}{3} = -17$$

The right answer is (c)

Question 2. The value of the integral $\int_{-1}^{1} 2|x|^3 \, dx$ is equal to :

(a) 2

(b) 1

(c) 0

(d) -1

Answer: Note that $|x| = x$ if $x \geq 0$, and $|x| = -x$ if $x < 0$.

$\int_{-1}^{1} 2|x|^3 \, dx = \int_{-1}^{0} 2|x|^3 \, dx + \int_{0}^{1} 2|x|^3 \, dx$

$= \int_{-1}^{0} 2(-x)^3 \, dx + \int_{0}^{1} 2x^3 \, dx = -\int_{-1}^{0} 2x^3 \, dx + \int_{0}^{1} 2x^3 \, dx$

$= -2\left[\frac{x^4}{4}\right]_{-1}^{0} + 2\left[\frac{x^4}{4}\right]_{0}^{1}$

$= -2\left[0 - \frac{1}{4}\right] + 2\left[\frac{1}{4} - 0\right] = \frac{1}{2} + \frac{1}{2} = 1$

The right answer is (b)

Question 3. The value of the integral $\int \frac{\sin(\tan(x))}{\cos^2(x)} \, dx$ is equal to :

(a) $\cos(\tan(x)) + c$

(b) $\sin(\tan(x)) + c$

(c) $-\cos(\tan(x)) + c$

(d) $-\sin(\tan(x)) + c$

Answer : $\int \frac{\sin(\tan(x))}{\cos^2(x)}\,dx = \int \sin(\tan(x))\sec^2(x)\,dx = -\cos(\tan(x)) + c$
The right answer (c)

Question 4. The derivative of the integral $\int_0^x \left[1 + \frac{d\tan(t)}{dt}\right]\,dt$ is equal to :

(a) $1 + \tan x$

(b) $1 - \tan x$

(c) $1 - \sec^2 x$

(d) $1 + \sec^2 x$

Answer : $\frac{d}{dx}\int_0^x \left[1 + \frac{d\tan(t)}{dt}\right]\,dt = 1 + \frac{d\tan x}{dx} = 1 + \sec^2 x$
The right answer is (d)

Question 5. If $G(x) = \int_e^{x^2} \frac{\ln(t)}{4}\,dt$ then $G'(e)$ is equal to :

(a) $2e$

(b) 1

(c) e

(d) $4e$

Answer : $G'(x) = \frac{d}{dx}\int_e^{x^2} \frac{\ln(t)}{4}\,dt = \frac{\ln(x^2)}{4}(2x)$
$= \frac{2\ln(x)}{4}(2x) = \frac{4x\ln(x)}{4} = x\ln(x)$
$G'(e) = e\ln(e) = e(1) = e$
The right answer is (c)

Question 6. If $\log_2\left(\frac{x-1}{x}\right) = 1$, then x is equal to :

(a) 1

(b) 2

(c) $\frac{1}{2}$

(d) -1

Answer : $\log_2\left(\frac{x-1}{x}\right) = 1 \Rightarrow \frac{x-1}{x} = 2 \Rightarrow x-1 = 2x \Rightarrow x = -1$
The right answer is (d)

Question 7. The value of the integral $\int_0^1 5^x \, dx$ is equal to :

(a) $\frac{4\ln 5}{5}$

(b) $\frac{\ln 5}{4}$

(c) $\frac{4}{\ln 5}$

(d) $\frac{5\ln 5}{4}$

Answer : $\int_0^1 5^x \, dx = \left[\frac{5^x}{\ln 5}\right]_0^1 = \frac{5}{\ln 5} - \frac{1}{\ln 5} = \frac{4}{\ln 5}$
The right answer is (c)

Question 8. The value of the integral $\int x\sqrt{x^2+1}\, dx$ is equal to :

(a) $\frac{1}{2}x^2\sqrt{x^2+1} + c$

(b) $\frac{2}{3}(x^2+1)^{\frac{3}{2}} + c$

(c) $-\frac{2}{3}(x^2+1)^{\frac{3}{2}} + c$

(d) $\frac{1}{3}(x^2+1)^{\frac{3}{2}} + c$

Answer : $\int x\sqrt{x^2+1}\, dx = \frac{1}{2}\int (x^2+1)^{\frac{1}{2}}(2x)\, dx = \frac{1}{2}\ \frac{(x^2+1)^{\frac{3}{2}}}{\frac{3}{2}} + c$
$= \frac{1}{3}(x^2+1)^{\frac{3}{2}} + c$
The right answer is (d)

Question 9. The value of the integral $\int_0^1 \frac{e^x}{(e^x+1)^2}\, dx$ is equal to :

(a) $\frac{e-1}{2(1+e)}$

(b) -1

(c) 0

(d) $\frac{1}{(1+e)^2}$

Answer : $\int_0^1 \frac{e^x}{(e^x+1)^2}\ dx = \int (e^x+1)^{-2}\ e^x\ dx$
$= \left[\frac{(e^x+1)^{-1}}{-1}\right]_0^1 = \left[\frac{-1}{e^x+1}\right]_0^1$
$= \frac{-1}{e+1} - \frac{-1}{1+1} = \frac{-1}{e+1} + \frac{1}{2} = \frac{-2+e+1}{2(e+1)} = \frac{e-1}{2(e+1)}$
The right answer is (a).

Question 10. The value of the integral $\int \frac{1}{\sqrt{16-25x^2}}\, dx$ is equal to :

(a) $-\frac{\cos^{-1}\left(\frac{x}{16}\right)}{25} + c$

(b) $\frac{\cos^{-1}\left(\frac{x}{16}\right)}{25} + c$

(c) $\frac{\sin^{-1}\left(\frac{5x}{4}\right)}{5} + c$

(d) $-\frac{\sin^{-1}\left(\frac{5x}{4}\right)}{5} + c$

Answer : $\int \frac{1}{\sqrt{16-25x^2}}\, dx = \frac{1}{5}\int \frac{5}{\sqrt{(4)^2-(5x)^2}}\, dx = \frac{1}{5}\sin^{-1}\left(\frac{5x}{4}\right) + c$
The right answer is (c)

Full questions

Question 11. Use **Trapizoidal rule** to approximate the integral $\int_1^3 \sqrt{x^2+3}\ dx$ with $n = 4$. [3 marks]

Answer : $[a,b] = [1,3]$, $n = 4$, and $(x) = \sqrt{3+x^2}$.
$\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2} = 0.5$
$x_0 = 1$, $x_1 = 1.5$, $x_2 = 2$, $x_3 = 2.5$, $x_4 = 3$.
$\int_1^3 \sqrt{3+x^2}\ dx \approx \frac{3-1}{2(4)}\left[f(1) + 2f(1.5) + 2f(2) + 2f(2.5) + f(3)\right]$
$\int_1^3 \sqrt{3+x^2}\ dx \approx \frac{1}{4}\left[2 + 2(2.29129) + 2(2.64575) + 2(3.04138) + 3.4641\right]$
$\int_1^3 \sqrt{3+x^2}\ dx \approx \frac{21.4209}{4} \approx 5.35523$

Question 12. If $f(x) = x^{\cosh x}$, then find $f'(x)$. [2 marks]

Answer : $f(x) = x^{\cosh x} \Rightarrow \ln|f(x)| = \ln|x|^{\cosh x} = \cosh x \ln|x|$
Differentiate both sides
$\frac{f'(x)}{f(x)} = \sinh x\ \ln|x| + \cosh x\ \left(\frac{1}{x}\right)$
$f'(x) = f(x)\left[\sinh x\ \ln|x| + \frac{\cosh x}{x}\right]$

$f'(x) = x^{\cosh x}\left[\sinh x \ \ln|x| + \dfrac{\cosh x}{x}\right]$

Question 13. Find the number z that satisfies the conclusion of the Mean Value Theorem for the function $f(x) = \cos(2x)$ where $x \in \left[0, \frac{\pi}{2}\right]$. And also find the averahe value f_{av} of $f(x)$. [3 marks]

Answer :

First : Calculating f_{av}

$$f_{av} = \frac{\int_0^{\frac{\pi}{2}} \cos 2x \ dx}{\frac{\pi}{2} - 0} = \frac{2}{\pi}\ \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos(2x)\ 2\ dx = \frac{1}{\pi}\left[\sin(2x)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{\pi}\left[\sin(\pi) - \sin(0)\right] = \frac{1}{\pi}(0) = 0\ .$$

Second : Calculating the number z that satisfies the conclusion of the MVT

According to MVT there exists a number $z \in \left[0, \frac{\pi}{2}\right]$ such that

$$\cos(2z) = \frac{\int_0^{\frac{\pi}{2}} \cos 2x \ dx}{\frac{\pi}{2} - 0}$$

$$\cos(2z) = 0 \Rightarrow 2z = \frac{\pi}{2} \Rightarrow z = \frac{\pi}{4}$$

Question 14. Evaluate the integral $J = \displaystyle\int \frac{\cos(x)}{\sin(x)\sqrt{4 - \sin^2(x)}}\, dx$ [2 marks]

Answer :

$$J = \int \frac{\cos(x)}{\sin(x)\sqrt{4 - \sin^2(x)}}\, dx = \int \frac{\cos(x)}{\sin(x)\sqrt{(2)^2 - (\sin(x))^2}}\, dx$$

$$= -\frac{1}{2} sech^{-1}\left(\frac{\sin x}{2}\right) + c$$
