King Saud University
Faculty of Sciences
Department of Mathematics

Midterm Exam-2	Math 106	November 2026

Question 1 : (3+3+3)

a) Find $\lim\limits_{x \longrightarrow +\infty} \left(\dfrac{x^2-1}{x^2} \right)^x$.

b) Compute the integral $\int x^4 (\ln x)^2 dx$.

c) Compute $\int \sin^4(x) dx$.

Question 2 : (3+3+3)

a) Compute $\int (\tan x)^5 (\sec x)^7 dx$.

b) Evaluate $\int \dfrac{dx}{x^2\sqrt{x^2+16}}$.

c) Find the indefinite integral $\int \dfrac{x^2}{x^2-3x+2} dx$.

Question 3 : (2+2+3)

a) Compute the indefinite integral $\int \dfrac{dx}{\sqrt{3-2x-x^2}}$.

b) Evaluate $\int \dfrac{dx}{5+4\cos x}$.

c) Does the integral $\int_e^{+\infty} \dfrac{4}{x(\ln x)^5} dx$ converge? Find its value if it does.

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Solution of the second examination Math 106 Semester I
November 2026

Question 1 : (3+3+3)

(a)

$$\begin{aligned}\lim_{x\to+\infty}\left(\frac{x^2-1}{x^2}\right)^x &= \lim_{x\to+\infty} e^{x\ln(1-\frac{1}{x^2})}\\ &\overset{t=\frac{1}{x}}{=} \lim_{t\to 0} e^{\frac{\ln(1-t^2)}{t}} \qquad (\mathbf{2})\\ &= 1 \text{ by l'Hopital} \qquad (\mathbf{1})\end{aligned}$$

(b) $\displaystyle\int x^4(\ln x)^2\,dx$

Integrate by parts with $u=(\ln x)^2$, $dv=x^4\,dx$, we get

$$I=\frac{x^5}{5}(\ln x)^2-\frac{2}{5}\int x^4\ln x\,dx. \qquad (\mathbf{1})$$

For $J=\int x^4\ln x\,dx$, we integrate also by parts with $u=\ln x$, $dv=x^4dx$, we get

$$J=\frac{x^5}{5}\ln x-\frac{1}{5}\int x^4\,dx=\frac{x^5}{5}\ln x-\frac{x^5}{25}.$$

Therefore

$$\int x^4(\ln x)^2\,dx=\frac{x^5}{5}(\ln x)^2-\frac{2x^5}{25}\ln x+\frac{2x^5}{125}+C. \qquad (\mathbf{2})$$

(c) $\displaystyle\int \sin^4 x\,dx$. By parts $u=\sin^3 x$ and $v'=\sin x$, we get

$$\int \sin^4 x\,dx=-\frac{1}{4}\cos x\sin^3 x+\frac{3}{4}\int \sin^2 x dx \qquad (\mathbf{1.5})$$

$\int \sin^2 x dx = \frac{x}{2} - \frac{1}{2}\sin x \cos x + C$. Then

$$\int \sin^4 x\, dx = -\frac{1}{4}\cos x \sin^3 x + \frac{3x}{8} - \frac{3}{8}\sin x \cos x + C \qquad (\mathbf{1.5})$$

We can also have (using $\sin^2 x = \frac{1-\cos(2x)}{2}$, $\cos^2 x = \frac{1+\cos(2x)}{2}$.)

$$\int \sin^4 x\, dx = \frac{3x}{8} - \frac{\sin(2x)}{4} + \frac{\sin(4x)}{32} + C$$

Question 2 : (3+3+3)

(a) $I = \int \tan^5 x \, \sec^7 x \, dx$

Let $u = \sec x$, $du = \sec x \tan x \, dx$:

$$I = \int (u^2-1)^2 u^6 \, du = \int \left(u^{10} - 2u^8 + u^6\right) du = \frac{u^{11}}{11} - \frac{2u^9}{9} + \frac{u^7}{7} + C. \quad (\mathbf{2})$$

Therefore

$$\int \tan^5 x \, \sec^7 x \, dx = \frac{\sec^{11} x}{11} - \frac{2\sec^9 x}{9} + \frac{\sec^7 x}{7} + C. \qquad (\mathbf{1})$$

(b) $\int \frac{dx}{x^2\sqrt{x^2+16}}$

Let $x = 4\tan\theta$, $dx = 4\sec^2\theta \, d\theta$, $\sqrt{x^2+16} = 4\sec\theta$:

$$\int \frac{4\sec^2\theta \, d\theta}{16\tan^2\theta \cdot 4\sec\theta} = \frac{1}{16}\int \frac{\sec\theta}{\tan^2\theta} \, d\theta = \frac{1}{16}\int \frac{\cos\theta}{\sin^2\theta} \, d\theta. \qquad (\mathbf{1.5})$$

With $u = \sin\theta$, $du = \cos\theta \, d\theta$:

$$\frac{1}{16}\int u^{-2} \, du = -\frac{1}{16u} + C = -\frac{1}{16\sin\theta} + C.$$

From $\tan\theta = x/4$: $\sin\theta = \frac{x}{\sqrt{x^2+16}}$. Hence

$$-\frac{1}{16\sin\theta} = -\frac{\sqrt{x^2+16}}{16x} + C.$$

$$\int \frac{dx}{x^2\sqrt{x^2+16}} = -\frac{\sqrt{x^2+16}}{16x} + C \qquad (\mathbf{1.5})$$

(c) $\displaystyle\int \frac{x^2}{x^2-3x+2}\,dx$

We have

$$\frac{x^2}{x^2-3x+2} = 1 + \frac{3x-2}{(x-1)(x-2)} = 1 - \frac{1}{x-1} + \frac{4}{x-2}. \qquad \textbf{(1.5)}$$

$$\int \left(1 - \frac{1}{x-1} + \frac{4}{x-2}\right) dx = x - \ln|x-1| + 4\ln|x-2| + C.$$

$$\int \frac{x^2}{x^2-3x+2}\,dx = x - \ln|x-1| + 4\ln|x-2| + C \qquad \textbf{(1.5)}$$

Question 3 : (2+2+3)

(a) $\displaystyle\int \frac{dx}{\sqrt{3-2x-x^2}} = \int \frac{dx}{\sqrt{4-(x+1)^2}} = \sin^{-1}\left(\frac{x+1}{2}\right) + C. \qquad \textbf{(1+1)}$

(b) $\displaystyle\int \frac{dx}{5+4\cos x}$

Use the substitution $t = \tan\frac{x}{2}$,

$$5 + 4\cos x = 5 + 4\cdot\frac{1-t^2}{1+t^2} = \frac{9+t^2}{1+t^2}.$$

Therefore

$$\int \frac{dx}{5+4\cos x} = \int \frac{1+t^2}{9+t^2}\cdot\frac{2\,dt}{1+t^2} = 2\int \frac{dt}{t^2+9} = \frac{2}{3}\tan^{-1}\left(\frac{t}{3}\right) + C. \qquad \textbf{(1+1)}$$

With $t = \tan\frac{x}{2}$:

$$\int \frac{dx}{5+4\cos x} = \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right) + C$$

(c) Convergence of $\displaystyle\int_e^{+\infty} \frac{4}{x(\ln x)^5}\,dx$

Substitute $u = \ln x$, $du = \dfrac{dx}{x}$.

$$\int_e^{+\infty} \frac{4}{x(\ln x)^5}\,dx = 4\int_1^{+\infty} u^{-5}\,du = 1.$$

The improper integral converges, and its value is 1. **(2+1)**