B) Linear differential equations of the first order

All equations that we can write in the form $y' + P(x)y = Q(x)$

<u>Examples:</u> ① $x^2 y' + y = 1$

$\Leftrightarrow y' + \frac{1}{x^2} y = \frac{1}{x^2}$

$P(x) = \frac{1}{x^2}$; $Q(x) = \frac{1}{x^2}$

② $y' + y = 1$

$P(x) = 1$; $Q(x) = 1$

③ $x^3 y' + (\ln x) y = 0$

$\Leftrightarrow y' + \frac{\ln x}{x^3} y = 0$

$P(x) = \frac{\ln x}{x^3}$; $Q(x) = 0$

④ $3x^2 y' = \ln x$

$\Leftrightarrow y' = \frac{\ln x}{3x^2}$

$P(x) = 0$; $Q(x) = \frac{\ln x}{3x^2}$

How to solve these equations:

Examples:

1. $y' + y = 1$

If we compute

$$(ye^x)' = y'e^x + ye^x$$
$$= (y'+y)e^x = e^x$$

$y' + y = 1 \iff (ye^x)' = e^x$

Then $\iff ye^x = \int e^x\,dx$

$\iff ye^x = e^x + C$

$\iff \boxed{y = 1 + Ce^{-x}}$

2. $y' + 2xy = x$

We compute:

$$(ye^{x^2})' = y'e^{x^2} + 2xe^{x^2}y$$
$$= (y' + 2xy)e^{x^2}$$
$$= xe^{x^2}$$

$y' + 2xy = x \Rightarrow (ye^{x^2})' = xe^{x^2}$

$\Rightarrow ye^{x^2} = \int xe^{x^2}dx = \frac{1}{2}\int e^u du$

$= \frac{1}{2}e^u + C$

$u = x^2$

$du = 2x\,dx$

$= \frac{1}{2}e^{x^2} + C$

$\Rightarrow \boxed{y = \frac{1}{2} + Ce^{-x^2}}$

In general:

$$y' + P(x)y = Q(x)$$

Consider $\boxed{\rho(x) = e^{\int P(x)dx}}$

$\rho'(x) = P(x)\,e^{\int P(x)dx}$

$= P(x)\rho(x)$

$\rho(x)$ is called an integrating factor

We have $(y\rho(x))' = y'\rho(x) + P(x)\rho(x)y$

$= (y' + P(x)y)\rho(x)$

$(y\rho(x))' = Q(x)\rho(x)$

$$y\,\rho(x) = \int Q(x)\,\rho(x)\,dx$$

$$\Rightarrow \boxed{y = \frac{1}{\rho(x)} \int Q(x)\,\rho(x)\,dx}$$

To solve ~~an~~ linear equation of the first order

$$y' + P(x)\,y = Q(x)$$

① we compute an integrating factor

$$\rho(x) = e^{\int P(x)\,dx}$$

② we compute the solution

$$y = \frac{1}{\rho(x)} \int \rho(x)\,Q(x)\,dx$$

In the ~~eq~~. example: $y' + y = 1$

$P(x) = 1$; $\rho(x) = e^{\int 1 dx} = e^{x}$

In the example : $y' + 2x\,y = x$

$P(x) = 2x$; $\rho(x) = e^{\int 2x dx} = e^{x^2}$

Example 3:

Find the general solution to the differential equation

$$x\frac{dy}{dx} - 2y = x^3 \sec x \tan x$$

and then find the particular solution which satisfies $y(\pi) = \pi^2$

Solution: We write the equation in the form:

$$y' - \frac{2}{x} y = \underbrace{x^2 \sec x \tan x}_{Q(x)}$$

We find an integrating factor

$$P(x) = -\frac{2}{x}$$

$$\rho(x) = e^{\int -\frac{2}{x} dx} = e^{-2\int \frac{1}{x} dx} = e^{-2\ln|x|}$$

$$= e^{\ln x^{-2}} = x^{-2} = \frac{1}{x^2}$$

The solution is

$$y = x^2 \int \frac{1}{x^2} x^2 \sec x \tan x \, dx$$

$$= x^2 \int \sec x \tan x \, dx = x^2(\sec x + C)$$

$$\boxed{y = x^2 \sec x + Cx^2}$$

For the particular solution which satisfies $y(\pi) = \pi^2$

$$\pi^2 = \pi^2 \underbrace{\sec \pi}_{-1} + C\pi^2$$

$$\Leftrightarrow \pi^2 = -\pi^2 + C\pi^2$$

$$\Leftrightarrow C\pi^2 = 2\pi^2 \Leftrightarrow \boxed{C = 2}$$

Then the particular solution is

$$\boxed{y = x^2 \sec x + 2x^2}$$

Example 4: The same question for

$$y' + 3y = e^{-2x}$$

Solution:

Here: $P(x) = 3$; $Q(x) = e^{-2x}$

An integrating factor:

$$\rho(x) = e^{\int 3dx} = e^{3x}$$

To solve:

$$y = \frac{1}{e^{3x}} \int e^{3x} \cdot e^{-2x} dx = \frac{1}{e^{3x}} \int e^{x} dx$$

$$= \frac{1}{e^{3x}} \left(e^{x} + C \right) = e^{-2x} + \frac{C}{e^{3x}}$$

$$= \frac{1}{e^{2x}} + \frac{C}{e^{3x}} = e^{-2x} + Ce^{-3x}$$

$$\boxed{y = e^{-2x} + Ce^{-3x}}$$

A particular solution for $y(0) = 0$

$0 = 1 + C \iff C = -1$

$$\boxed{y = e^{-2x} - e^{-3x}}$$