Differential Equations

① A differential equation is an equation involving at least one of the derivatives of an unknown function

Examples: ① $y'' + x = e^x$

② $y' + y = 1$

③ $y^{(3)} = 1$

Find all functions y.

② The order of the differential equation

Order is the highest derivative involved in the equation.

Examples: ① $y' + y^{(3)} - xy = 1$

Order $= 3$

② $y'' + y = 1$ Order $= 2$

③ $y' y'' y^{(3)} = 1$

The order is 3.

③ Linear differential equations

These are equations of the form:

$$a_n(x)\, y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \cdots + a_2(x)\, y'' + a_1(x)\, y' + a_0(x)\, y = b(x)$$

Examples:

$$2x\, y'' + x\, y' - x^2 y = 3e^x$$

$$x^2 e^x y' + \frac{1}{\sqrt{1+x^2}}\, y = 2\cos x$$

$$e^{x^2} y' = 3\cos(x^2)$$

In this lecture we are going to study differention equation of the first order

(1) Method of separation of variables

If we can write a differential equation of the first order in the form: $M(x) + N(y)y' = 0$ we can apply this method.

Examples: $3y^2 e^x + \frac{dy}{dx} = 0$

$$\Leftrightarrow \underbrace{3e^x}_{M(x)} + \underbrace{\frac{1}{y^2}}_{N(y)} \frac{dy}{dx} = 0$$

How to solve it:

$$M(x) + N(y)\, y' = 0$$

$$\Leftrightarrow \quad M(x) = -N(y)\, y'$$

$$\Leftrightarrow \quad \int M(x)\,dx = -\int N(y)\, \underbrace{y'\,dx}_{dy} = -\int N(y)\,dy$$

Example: $3e^x + \frac{1}{y^2}\frac{dy}{dx} = 0$

Then $\int 3e^x\,dx + \int \frac{1}{y^2}\,dy = C$

$\Rightarrow \quad 3e^x + \left(-\frac{1}{y}\right) = C$

$\Rightarrow \quad \frac{1}{y} = 3e^x - C$

$\Rightarrow \quad \boxed{y = \frac{1}{3e^x - C}}$

Example 2:

$\cos^2 x\,dy - y^2\,dx = 0$

$\Rightarrow \quad \frac{1}{y^2}\,dy - \frac{1}{\cos^2 x}\,dx = 0$

$\Rightarrow \quad \int \frac{1}{y^2}\,dy - \int \frac{1}{\cos^2 x}\,dx = C$

$\Rightarrow \quad -\frac{1}{y} - \tan x = C$

$\Rightarrow \quad \frac{1}{y} = -\tan x - C$

$$\Rightarrow \boxed{y = \frac{1}{-\tan x - C}}$$

<u>Example:</u> $x^2 dy + y^2 dx = 0$

$$\Rightarrow \frac{1}{y^2} dy + \frac{1}{x^2} dx = 0$$

$$\Rightarrow \int \frac{1}{y^2} dy + \int \frac{1}{x^2} dx = C$$

$$\Rightarrow -\frac{1}{y} - \frac{1}{x} = C$$

$$\Rightarrow \frac{1}{y} = -C - \frac{1}{x} = \frac{-Cx - 1}{x}$$

$$\Rightarrow \boxed{y = \frac{x}{-Cx - 1}}$$

<u>Example:</u> $\sqrt{1+x^2}\, y' + x(1+y) = 0$

$$\Rightarrow \frac{y'}{1+y} + \frac{x}{\sqrt{1+x^2}} = 0$$

$$\Rightarrow \int \frac{1}{1+y} dy + \int \frac{\frac{1}{2} 2x}{\sqrt{1+x^2}} dx = C$$

By substitution $u = 1 + x^2$, $du = 2x\, dx$

$$\Leftrightarrow \ln|1+y| + \int \frac{\frac{1}{2}}{\sqrt{u}}\, du = C$$

$$\Leftrightarrow \ln|1+y| + \sqrt{u} = C$$

$$\Leftrightarrow \ln|1+y| + \sqrt{1+x^2} = C \quad \leftarrow$$

$$\Leftrightarrow \ln|\cancel{1}+y| = C - \sqrt{1+x^2}$$

$$\Leftrightarrow |1+y| = e^{C - \sqrt{1+x^2}}$$

$$\Leftrightarrow 1+y = \pm\, e^{C - \sqrt{1+x^2}}$$

$$\Leftrightarrow \boxed{y = \pm\, e^{C - \sqrt{1+x^2}} - 1} \quad \leftarrow$$

$$y = K e^{-\sqrt{1+x^2}} - 1$$