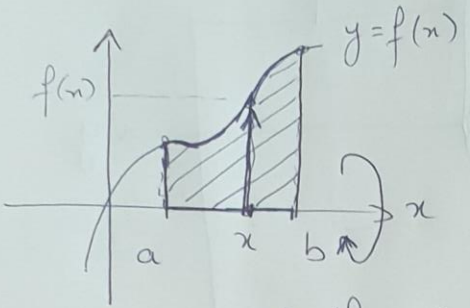
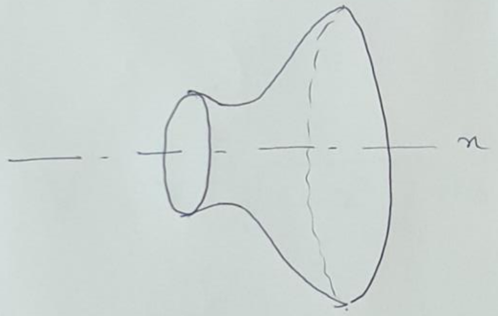
The volumes of revolutions

① The disk method طريقة الأقراص الدائرية

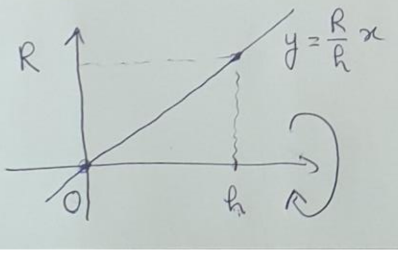


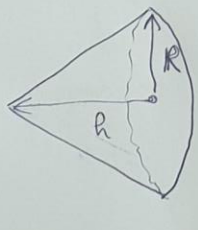
Let $f(x) \geq 0$ on $[a,b]$ and we would lik to compute the volume of the region bounded $y = f(x)$; $y = 0$; $x = a$; $x = b$ rotated about the x-axis



$$V = \int_a^b \pi f(x)^2 \, dx$$

Examples:



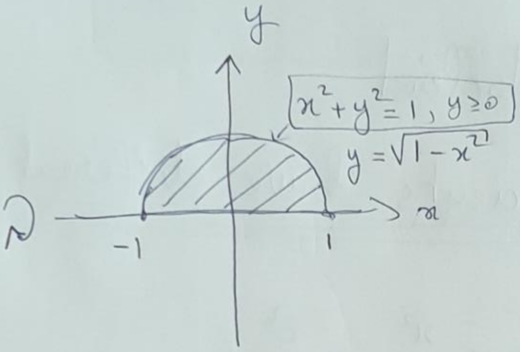


$V = ?$

$$\boxed{V} = \int_0^h \pi \left(\frac{R}{h}x\right)^2 dx = \pi \frac{R^2}{h^2}\int_0^h x^2 dx = \frac{\pi R^2}{h^2}\left[\frac{x^3}{3}\right]_0^h$$

$$= \frac{\pi R^2}{3h^2}\left(h^3 - 0\right) = \boxed{\frac{\pi R^2 h}{3}}$$

Example:



$$V = \pi\int_{-1}^{1}\left(\sqrt{1-x^2}\right)^2 dx = \pi\int_{-1}^{1}(1-x^2)\,dx$$

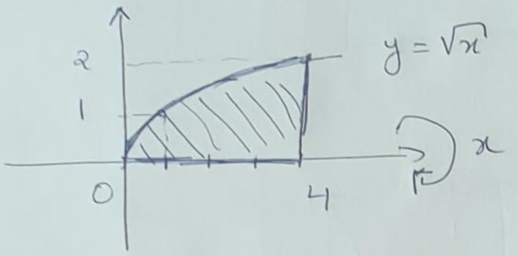
$$= \pi\left[x - \frac{x^3}{3}\right]_{-1}^{1} = \pi\left[\left(1-\frac{1}{3}\right) - \left(-1+\frac{1}{3}\right)\right]$$

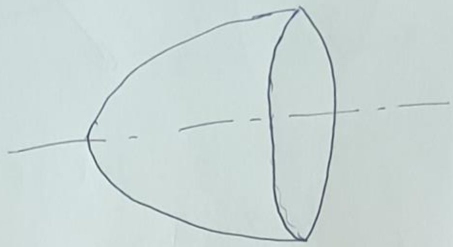
$$= \frac{4\pi}{3}$$

The volume of a ball or

is $\boxed{V = \frac{4\pi}{3} R^2}$

Example:



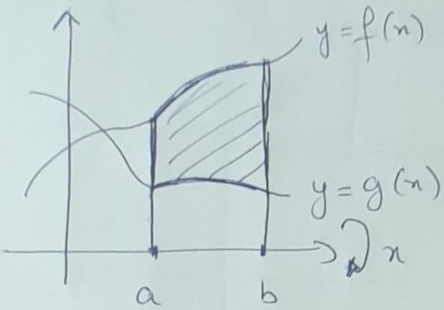


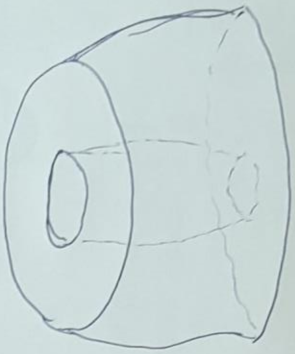
The volume of the region bounded by $y = \sqrt{x}$; $y = 0$, $x = 4$ rotated about the x-axis is given by:

$$V = \pi \int_0^4 (\sqrt{x})^2 \, dx = \pi \int_0^4 x \, dx = \pi \left[\frac{x^2}{2} \right]_0^4$$

$$= 8\pi .$$

② The washer method:

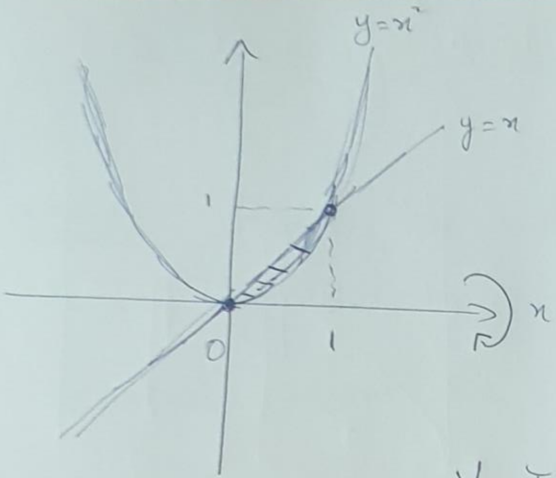




Let $f(x) \geq g(x)$ on $[a, b]$. The volume of the region bounded by $y = f(x)$, $y = g(x)$; $x = a$; $x = b$ rotated about the x-axis is given

$$V = \pi \int_a^b \left[f(x)^2 - g(x)^2 \right] dx$$

Examples: Compute the volume of the region bounded by $y = x^2$, $y = x$ rotated about the x-axis.



We solve

$$\begin{cases} y=x^2 \\ y=x \end{cases} \iff \begin{cases} y=x \\ x^2-x=0 \\ x(x-1)=0 \end{cases}$$

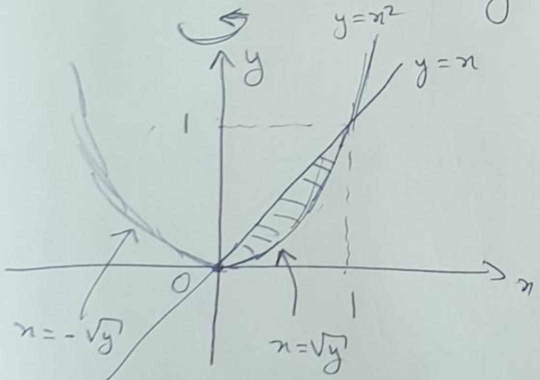
$$\iff \begin{cases} x=0\ ;\ y=0 \\ \text{or} \\ x=1\ ;\ y=1 \end{cases}$$

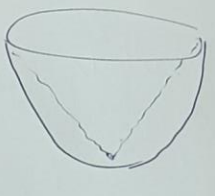
$$V = \pi \int_0^1 (x^2 - x^4)\,dx$$

$$= \pi \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1 = \pi \left(\frac{1}{3} - \frac{1}{5} \right)$$

$$\boxed{V = \frac{2\pi}{15}}$$

Example: The volume of the same region rotated about the y-axis

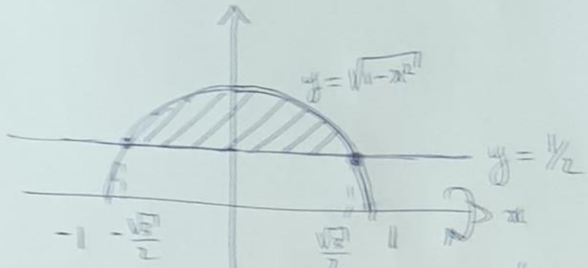


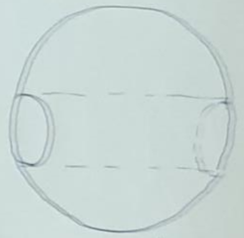


$$V = \pi \int_0^1 \left[(\sqrt{y})^2 - y^2 \right] dy = \pi \int_0^1 (y - y^2)\,dx = \pi \left[\frac{y^2}{2} - \frac{y^3}{3} \right]_0^1$$

$$= \pi\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\pi}{6} \qquad \boxed{V = \frac{\pi}{6}}$$

Example: Find the volume of ~~rota~~ the region bounded by $y = \sqrt{1-x^2}$, $y = \frac{1}{2}$ rotated about the x-axis





we solve:
$$\begin{cases} y = 1/2 \\ y = \sqrt{1-x^2} \end{cases} \iff \begin{cases} y = 1/2 \\ \frac{1}{2} = \sqrt{1-x^2} \end{cases}$$

$$\iff \begin{cases} y = 1/2 \\ \frac{1}{4} = 1 - x^2 \end{cases} \iff \begin{cases} y = 1/2 \\ x^2 = \frac{3}{4} \end{cases} \iff \begin{cases} x = \frac{\sqrt{3}}{2};\ y = 1/2 \\ \text{or} \\ x = -\frac{\sqrt{3}}{2},\ y = 1/2 \end{cases}$$

$$V = \pi \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \left[\left(\sqrt{1-x^2}\right)^2 - \left(\frac{1}{2}\right)^2\right] dx = \pi \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \left(\frac{3}{4} - x^2\right) dx$$

$$= \pi \left[\frac{3}{4}x - \frac{x^3}{3}\right]_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} = \pi \left[\underbrace{\left(\frac{3\sqrt{3}}{8} - \frac{\sqrt{3}}{8}\right)}_{\frac{\sqrt{3}}{4}} - \underbrace{\left(-\frac{3\sqrt{3}}{8} + \frac{\sqrt{3}}{8}\right)}_{+\frac{\sqrt{3}}{4}}\right]$$

$$\boxed{V = \frac{\pi\sqrt{3}}{2}}$$