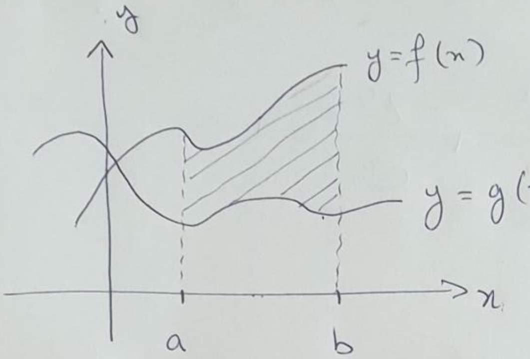
The area between two curves



$f(x) \geq g(x)$ on $[a, b]$

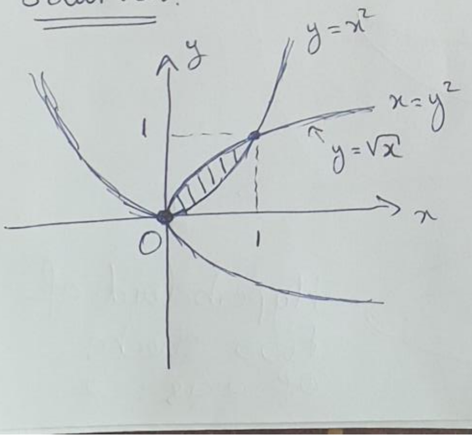
The area bounded by the region $y = f(x)$, $y = g(x)$, $x = a$, $x = b$

The area is equal to $\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

$$\int_a^b \big(f(x) - g(x)\big)\,dx$$

Example: Find the area of the region bounded by the curves $y = x^2$ and $x = y^2$

Solution:



We solve

$$\begin{cases} y = x^2 \\ x = y^2 \end{cases} \Leftrightarrow \begin{cases} y = x^2 \\ x = x^4 \end{cases} \Leftrightarrow \begin{cases} y = x^2 \\ x^4 - x = 0 \end{cases}$$

$$x^4 - x = x(x^3 - 1) = x(x-1)(x^2 + x + 1)$$

$\Delta = 1^2 - 4 = -3 < 0$

There is no root

$$\begin{array}{r|l} x^3 - 1 & x - 1 \\ \hline x^3 - x^2 & x^2 + x + 1 \\ x^2 - 1 & \\ x^2 - x & \\ x - 1 & \\ x - 1 & \\ 0 & \end{array}$$

Then $x = 0$ or $x = 1$

$y = 0$ $y = 1$

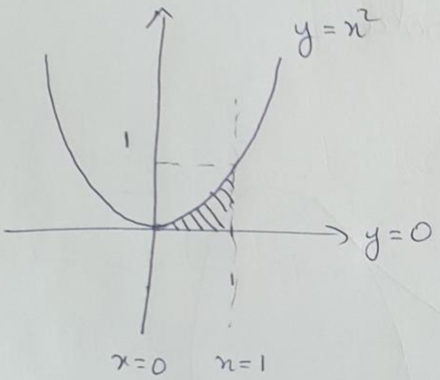
The area is equal to:

$$\int_0^1 (\underset{x^{1/2}}{\underset{\shortparallel}{\sqrt{x}}} - x^2)\,dx = \left[\frac{x^{3/2}}{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1 = \left(\frac{2}{3} - \frac{1}{3}\right) - 0$$

$$= \frac{1}{3}.$$

The area of the region bounded by:

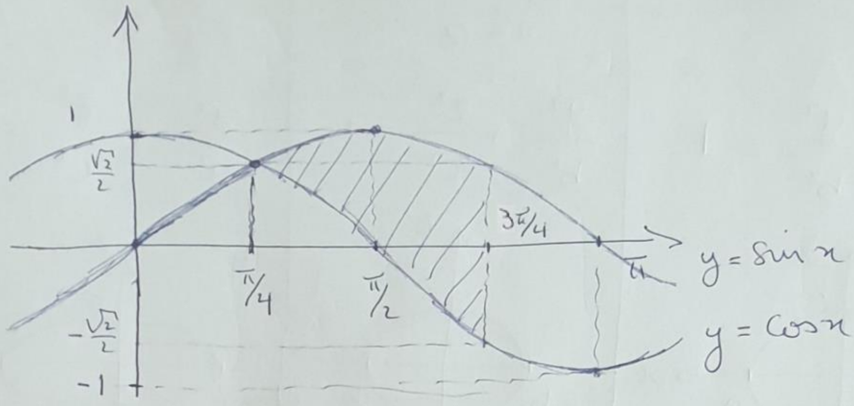
$y = x^2$; $y = 0$; $x = 0$; $x = 1$



$$\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$$

Example: Compute the area of the region bounded by $y = \cos x$; $y = \sin x$

$x = \frac{\pi}{4}$; $x = \frac{3\pi}{4}$.

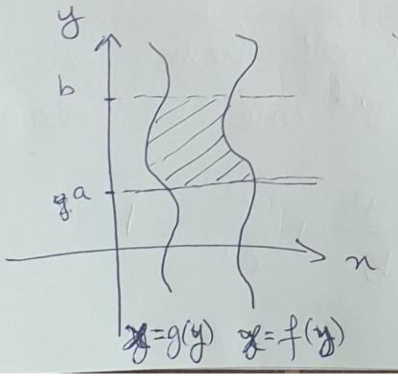


The area is equal to:

$$\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (\sin x - \cos x)\, dx = \Big[-\cos x - \sin x\Big]_{\pi/4}^{\frac{3\pi}{4}}$$

$$= \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) = \sqrt{2}$$

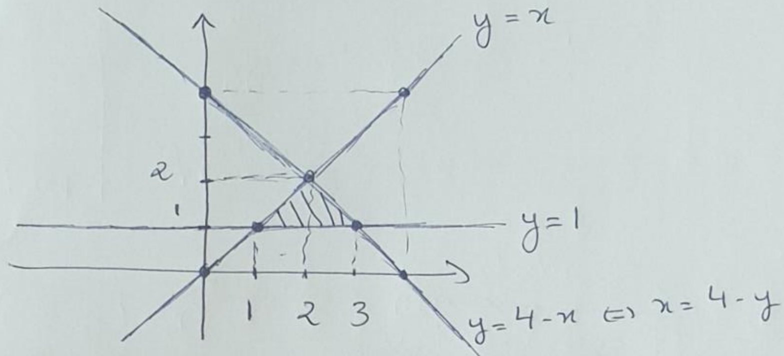
The area bounded by $x = f(y)$, $x = g(y)$; $y = a$, $y = b$ when $f(y) \geq g(y)$ on $[a, b]$



is given by

$$\int_a^b \big(f(y) - g(y)\big)\, dy$$

Example: Find the area of the region bounded by $y = x$; $y = 1$, $y = 4 - x$



The intersection point between $y = x$, $y = 1$:

We solve: $\begin{cases} y = 1 \\ y = x \end{cases} \Leftrightarrow \begin{cases} y = 1 \\ x = 1 \end{cases}$ $(1, 1)$

" " " $y = x$, $y = 4 - x$

We solve: $\begin{cases} y = x \\ y = 4 - x \end{cases} \Leftrightarrow \begin{cases} y = x \\ x = 4 - x \end{cases} \Leftrightarrow \begin{cases} y = 2 \\ x = 2 \end{cases}$

$(2, 2)$

" " " $y = 4 - x$; $y = 1$

we solve $\begin{cases} y = 1 \\ y = 4 - x \end{cases} \Leftrightarrow \begin{cases} y = 1 \\ 1 = 4 - x \end{cases} \Leftrightarrow \begin{cases} y = 1 \\ x = 3 \end{cases}$ $(3, 1)$

We compute it in two methods:

1. with respect to x: $(y = f(x) ; y = g(x))$

$$\text{Area} = \int_1^2 (x - 1)\,dx + \int_2^3 (4 - x - 1)\,dx = \left[\frac{x^2}{2} - x\right]_1^2 + \left[3x - \frac{x^2}{2}\right]_2^3$$

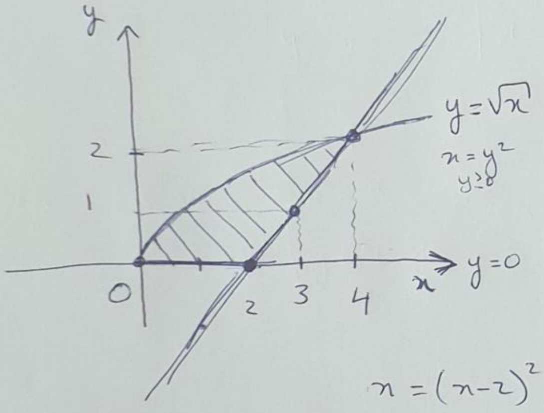
$$= \left((2-2) - \left(\tfrac{1}{2} - 1\right)\right) + \left(\underbrace{\left(9 - \tfrac{9}{2}\right)}_{\frac{9}{2}} - \underbrace{(6-2)}_{4}\right)$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

(2) with respect to y:

$$\text{Area} = \int_1^2 \left[(4-y) - 2y\right] dy = \left[4y - y^2\right]_1^2 = 4 - 3 = 1$$

Example: Find the area of the region bounded by $y = \sqrt{x}$; $y = 0$; $y = x - 2$.



$y = x - 2 \Leftrightarrow x = y + 2$

$$\begin{cases} y = \sqrt{x} \\ y = 0 \end{cases} \Leftrightarrow (x,y) = (0,0)$$

$$\begin{cases} y = x - 2 \\ y = 0 \end{cases} \Leftrightarrow (x,y) = (2,0)$$

$$\begin{cases} y = \sqrt{x} \\ y = x - 2 \end{cases} \Rightarrow \begin{cases} y = \sqrt{x} \\ x = (x-2)^2 \end{cases}$$

$$x = (x-2)^2 \Rightarrow x^2 - 5x + 4 = 0$$

$$(x-1)(x-4) = 0$$

$$\cancel{x=1} \qquad y = 2 \checkmark$$

$$(x,y) = (4,2).$$

We compute the area in two methods:

(1) with respect to x:

$$\text{Area} = \int_0^2 \sqrt{x}\, dx + \int_2^4 \left(\sqrt{x} - (x-2)\right) dx = \left[\frac{2x^{3/2}}{3}\right]_0^2 + \left[\frac{2x^{3/2}}{3} - \frac{x^2}{2} + 2x\right]_2^4$$

$$= \frac{\cancel{4\sqrt{2}}}{\cancel{3}} + \left(\frac{16}{3} - \cancel{8} + \cancel{8}\right) - \left(\frac{\cancel{4\sqrt{2}}}{\cancel{3}} - 2 + 4\right) = \boxed{\frac{10}{3}}$$

(2) with respect to y:

$$\text{Area} = \int_0^2 \left[(y+2) - y^2\right] dy = \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_0^2$$

$$= \left(2 + 4 - \frac{8}{3}\right) - 0 = \boxed{\frac{10}{3}}$$