Example:

Compute $\int \frac{6x+3}{x^2+2x+3}\,dx$

1st step: $1 = \deg P < \deg Q = 2$

we skip this step.

2nd step: x^2+2x+3

$3, 1, -1, -3$

$\begin{array}{cccc} 18 & 6 & 2 & 6 \\ \neq & \neq & \neq & \neq \\ 0 & 0 & 0 & 0 \end{array}$

The discriminant:

ax^2+bx+c

How to find the roots:

We compute the discriminant:

$\Delta = b^2 - 4ac$

If $\Delta > 0$	If $\Delta = 0$	If $\Delta < 0$
We have two different roots $x_1 = \frac{-b+\sqrt{\Delta}}{2a}$ $x_2 = \frac{-b-\sqrt{\Delta}}{2a}$	We have one double root $x_{1,2} = -\frac{b}{2a}$	We have no real roots

In our example:

x^2+2x+3

$\Delta = 2^2 - 4\times1\times3 = -8 < 0$

Another example:

$$x^2 - x - 1$$

$$\begin{array}{cc} 1 & -1 \\ -1 & 1 \\ \neq 0 & \neq 0 \end{array}$$

$$\Delta = (-1)^2 - 4 \times 1 \times (-1) = 5 > 0$$

We have two roots:

$$x_1 = \frac{1+\sqrt{5}}{2} \quad ; \quad x_2 = \frac{1-\sqrt{5}}{2}$$

Then $x^2 - x - 1 = \left(x - \frac{1+\sqrt{5}}{2}\right)\left(x - \frac{1-\sqrt{5}}{2}\right)$

We check:

$$\left(x - \frac{1+\sqrt{5}}{2}\right)\left(x - \frac{1-\sqrt{5}}{2}\right) = x^2 - \frac{1+\sqrt{5}}{2}x - \frac{1-\sqrt{5}}{2}x + \frac{(1+\sqrt{5})}{2} \cdot \frac{(1-\sqrt{5})}{2}$$

$$= x^2 - x + \frac{1 + \cancel{\sqrt{5}} - \cancel{\sqrt{5}} - 5}{4} = x^2 - x - 1$$

Back to our example $x^2 + 2x + 3$

$\Delta < 0$

We apply the method of completing the square : $x^2 + 2x + 3 = x^2 + 2x + 1 + 2$

$$= (x+1)^2 + 2.$$

3rd step:

$$\frac{6x+3}{x^2+2x+3}$$

We have $(x^2+2x+3)' = 2x+2$

Therefore:

$$\begin{array}{r|l} & \ 3 \\ \hline 2x+2 & 6x+3 \\ & 6x+6 \\ \hline & \ -3 \end{array}$$

$$6x+3 = 3(2x+2) - 3$$

Hence $\displaystyle\int \frac{6x+3}{x^2+2x+3}\,dx = 3\int \frac{2x+2}{x^2+2x+3}\,dx - 3\int \frac{1}{x^2+2x+3}\,dx$

$$= 3\ln(x^2+2x+3) - 3\int \frac{1}{x^2+2x+3}\,dx$$

$\displaystyle\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C$

We have:

$$\int \frac{1}{x^2+2x+3}\,dx = \int \frac{1}{(x+1)^2+2}\,dx$$

$$= \frac{1}{2}\int \frac{1}{1+\left(\frac{x+1}{\sqrt{2}}\right)^2}\,dx$$

By substitution: Let $u = \frac{x+1}{\sqrt{2}}$. Then $du = \frac{1}{\sqrt{2}}\,dx$

Therefore: $\frac{1}{2}\int \frac{1}{1+\left(\frac{x+1}{\sqrt{2}}\right)^2}\,dx = \frac{1}{2}\int \frac{\sqrt{2}}{1+u^2}\,du$

$$= \frac{\sqrt{2}}{2}\tan^{-1}u + C$$

$$= \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + C$$

Therefore:

$$\int \frac{6x+3}{x^2+2x+3}\,dx = 3\ln\left(x^2+2x+3\right) - \frac{3\sqrt{2}}{2}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + C$$

Example:

Compute $\int \frac{4x+1}{x^2+4x+8}\,dx$

1st step: We skip

2nd step: x^2+4x+8

$\Delta = 4^2 - 4\times1\times8 = -16 < 0$ No roots.

$x^2+4x+8 = x^2 + 2\times2x + 2^2 + 4$

$= (x+2)^2 + 4.$

We compute:

$$\int \frac{4x+1}{x^2+4x+8}\,dx \stackrel{?}{=}$$

$$\begin{array}{r|l} & 2 \\ \hline 2x+4 & 4x+1 \\ & 4x+8 \\ \hline & \quad -7 \end{array}$$

$$4x+1 = 2(2x+4) - 7$$

Therefore

$$\int \frac{4x+1}{x^2+4x+8}\,dx = 2\int \frac{2x+4}{x^2+4x+8}\,dx - 7\int \frac{1}{x^2+4x+8}\,dx$$

$$= 2\ln(x^2+4x+8) - 7\int \frac{1}{x^2+4x+8}\,dx$$

<u>But</u>: $\displaystyle\int \frac{1}{x^2+4x+8}\,dx = \int \frac{1}{(x+2)^2+4}\,dx$

$$= \frac{1}{4}\int \frac{1}{1+\left(\frac{x+2}{2}\right)^2}\,dx$$

Let $u = \dfrac{x+2}{2}$. Then $du = \frac{1}{2}\,dx$

Hence $\displaystyle\frac{1}{4}\int \frac{1}{1+\left(\frac{x+2}{2}\right)^2}\,dx = \frac{1}{2}\int \frac{1}{1+u^2}\,du$

$$= \frac{1}{2}\tan^{-1} u + C$$

$$= \frac{1}{2} \tan^{-1}\left(\frac{x+2}{2}\right) + C$$

Hence

$$\int \frac{4x+1}{x^2+4x+8}\, dx = 2\ln\left(x^2+4x+8\right) - \frac{7}{2}\tan^{-1}\left(\frac{x+2}{2}\right) + C$$