Example

Compute $\displaystyle\int \frac{x^3-1}{x^5+2x^4+x^3}\,dx$

$\displaystyle\int \frac{P(x)}{Q(x)}\,dx$

1st step: $3 = \deg P < \deg Q = 4$

we skip the 1st step.

2nd step: $x^5 + 2x^4 + x^3 = x^3\left(x^2+2x+1\right)$

$$= x^3(x+1)^2$$

3rd step:

$$\frac{x^3-1}{x^5+2x^4+x^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+1} + \frac{E}{(x+1)^2}$$

We have

$$x^3 - 1 = A\,x^2(x+1)^2 + B\,x(x+1)^2 + C(x+1)^2 + D\,x^3(x+1)$$
$$+ E\,x^3$$

For $x = 0$: $\quad C = -1$

For $x = -1$: $\quad -E = -2 \iff E = 2$

$$\underbrace{x^3 - 1 + (x+1)^2 - 2x^3}_{} = A\,x^2(x+1)^2 + B\,x(x+1)^2$$
$$+ D\,x^3(x+1)$$

$$-\cancel{3}x^3 + x^2 + 2x = (\cancel{3}x^2 + x + 2)x = (x+1)(-x+2)\,x$$

Then $\quad -x+2 = Ax(x+1) + B(x+1)^2 + Dx^2$

For $x=0$: $\quad B = 2$

For $x=-1$: $\quad D = 3$

Then we have:

$$-x+2 - 2(x+1)^2 - 3x^2 = Ax(x+1)$$

$$\Leftrightarrow \quad \underbrace{-x} + \cancel{2} - \underbrace{2x^2} - \underbrace{4x} - \cancel{2} - \underbrace{3x^2} = Ax(x+1)$$

$$\Leftrightarrow \quad -5x^2 - 5x$$

$$-5x(x+1) = Ax(x+1)$$

$$A = -5$$

So

$$\int \frac{x^3-1}{x^5+2x^4+x^3}\,dx = \int \frac{-5}{x}\,dx + \int \underset{\substack{\shortparallel \\ 2x^{-2}}}{\frac{2}{x^2}}\,dx + \int \underset{\substack{\shortparallel \\ -x^{-3}}}{\frac{-1}{x^3}}\,dx$$

$$+ \int \frac{3}{x+1}\,dx + \int \frac{2}{(x+1)^2}\,dx$$

$$= -5\ln|x| - 2\frac{1}{x} + \frac{1}{2}\frac{1}{x^2} + 3\ln|x+1|^2$$

$$- \frac{2}{x+1} + C$$

Example:

Compute $\displaystyle\int \frac{x^4+1}{x^5-x^3}\,dx$

1st step: we skip it $\quad 4<5$

2nd step: $x^5 - x^3 = x^3(x^2-1) = x^3(x+1)(x-1)$

3rd step: we have

$$\frac{x^4+1}{x^5-x^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+1} + \frac{E}{x-1}$$

Therefore:

$$x^4+1 = Ax^2(x+1)(x-1) + Bx(x+1)(x-1) + C(x+1)(x-1) + Dx^3(x-1) + Ex^3(x+1)$$

For $x=0$: $\quad -C = 1 \iff C=-1$

For $x=-1$: $\quad 2D = 2 \iff D = 1$

For $x=1$: $\quad 2E = 2 \iff E = 1$

Therefore:

$$Ax^2(x+1)(x-1) + Bx(x+1)(x-1) = x^4+1+(x+1)(x-1) - x^3(x-1) - x^3(x+1)$$

$$= x^4 + \cancel{1} + x^2 - \cancel{1} - x^4 + \cancel{x^3} - x^4 - \cancel{x^3}$$

$$= -x^4 + x^2 = x^2(-x^2+1)$$

$$= -x^2(x-1)(x+1)$$

$$(Ax+B)\,x(x+1)(x-1) = -x^2(x+1)(x-1)$$

Then $Ax+B = -x$

Hence : $A = -1$

$B = 0$

Therefore:

$$\int \frac{x^4+1}{x^5-x^3}\,dx = \int \frac{-1}{x}\,dx + \int \underset{\underset{-x^{-3}}{\shortparallel}}{\frac{-1}{x^3}}\,dx + \int \frac{1}{x+1}\,dx$$

$$+ \int \frac{1}{x-1}\,dx$$

$$= -\ln|x| + \frac{1}{2}\frac{1}{x^2} + \ln|x+1| + \ln|x-1| + C$$

Example: Compute $\displaystyle\int \frac{x^2+3}{x^3+2x}\,dx$

1st step : we skip

2nd step : $x^3+2x = x(\underbrace{x^2+2}_{\geq 2})$

No factorization

3rd step: $\dfrac{x^2+3}{x^3+2x} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+2}$

We have:

$$x^2 + 3 = A(x^2+2) + (Bx+C)x$$

For $x = 0$: $2A = 3 \iff A = 3/2$.

We have:

$$x^2 + 3 = Ax^2 + 2A + Bx^2 + Cx$$

$$= (A+B)x^2 + Cx + 2A.$$

Then $\begin{cases} A+B = 1 \\ C = 0 \\ 2A = 3 \end{cases} \iff \begin{cases} A = 3/2 \\ B = -1/2 \\ C = 0 \end{cases}$

Therefore:

$$\int \frac{x^2+3}{x^3+2x}\,dx = \int \frac{3/2}{x}\,dx + \int \underbrace{\frac{-\frac{1}{2}x}{x^2+2}}_{-\frac{1}{4}\frac{2x}{x^2+2}}\,dx$$

$$= \frac{3}{2}\ln|x| - \frac{1}{4}\ln(x^2+2) + C$$