Integration of rational functions

تكامل الدوال الكسرية

Example:

$$(2x+1) + \frac{3}{x-1} - \frac{1}{x+2}$$

$$= \frac{(2x+1)(x-1)(x+2) + 3(x+2) - (x-1)}{(x-1)(x+2)}$$

$$= \frac{(2x+1)(x^2+x-2) + 3x + 6 - x + 1}{x^2+x-2}$$

$$= \frac{2x^3 + x^2 + 2x^2 + x - 4x - 2 + 2x + 7}{x^2+x-2}$$

$$= \frac{2x^3+3x^2-x+5}{x^2+x-2}$$

Compute: $\displaystyle\int \frac{2x^3+3x^2-x+5}{x^2+x-2}\,dx$

we have:

$$\int \frac{2x^3+3x^2-x+5}{x^2+x-2}\,dx = \int \left(2x+1 + \frac{3}{x-1} - \frac{1}{x+2}\right) dx$$

$$= x^2 + x + 3\ln|x-1| - \ln|x+2| + C$$

How to deal with rational functions:

1st Step: For $\dfrac{P(x)}{Q(x)}$ P, Q polynomials.

If $\deg P(x) \geq \deg Q(x)$ we perform this step. If not we go to the next step.

We perform the Euclidean division

$$
\begin{array}{r|l}
 & 2x+1 \\
\hline
x^2+x-2 & 2x^3+3x^2-x+5 \\
 & -2x^3-2x^2+4x+0 \\
\cline{2-2}
 & 0+x^2+3x+5 \\
 & x^2+x-2 \\
\cline{2-2}
 & 0+2x+7
\end{array}
$$

This means:

$$2x^3+3x^2-x+5=(2x+1)(x^2+x-2)+(2x+7)$$

$$\Leftrightarrow \quad \frac{2x^3+3x^2-x+5}{x^2+x-2}=2x+1+\frac{2x+7}{x^2+x-2}$$

Example:

Compute $\displaystyle\int \frac{x^3-2x^2+x-3}{x^3-x}\,dx$

1st step: $\deg P = 3$; $\deg Q = 3$

they are equal

we apply step 1

$$\begin{array}{r|l} & 1 \\ \hline x^3-x & x^3-2x^2+x-3 \\ & x^3 \qquad\quad -x \\ \hline & -2x^2+2x-3 \end{array}$$

So $\displaystyle\frac{x^3-2x^2+x-3}{x^3-x} = 1 + \frac{-2x^2+2x-3}{x^3-x}$

Example: Compute $\displaystyle\int \frac{x+1}{x^2-3x+2}\,dx$

1st step: $\deg P = 1$; $\deg Q = 2$

$1 < 2$ we skip this step

2^{nd} Step: We make a factorization of the denominator:

In the 1^{st} example:

The denominator: $Q(x) = x^2 + x - 2$

$\cancel{2}, 1, -2, -1$

$4 \quad 0 \quad 0 \quad -2$

$\neq \quad ✓ \quad ✓ \quad \neq$

$0 \qquad\qquad\quad 0$

$Q(x) = (x-1)(x+2)$

In the 2^{nd} example:

$Q(x) = x^3 - x = x(x^2 - 1) = x(x-1)(x+1)$

$1, -1$

$0 \quad 0$

$✓$

In the 3^{rd} example:

$Q(x) = x^2 - 3x + 2 = (x-2)(x-1)$

$2, 1, -1, -2$

$0 \quad 0 \quad 6 \quad 12$

$✓ \quad ✓ \quad \neq \quad \neq$

$\qquad\quad 0 \quad\; 0$

3^{rd} Step:

1^{st} example: $$\frac{2x+7}{x^2+x-2} = \frac{A}{x-1} + \frac{B}{x+2}$$

توحيد المقام :

$$\frac{A}{x-1} + \frac{B}{x+2} = \frac{A(x+2)+B(x-1)}{(x-1)(x+2)}$$

$$= \frac{(A+B)x + (2A-B)}{x^2+x-2} = \frac{2x+7}{x^2+x-2}$$

$$\Leftrightarrow \begin{cases} A+B = 2 \\ 2A-B = 7 \end{cases} \Leftrightarrow \begin{cases} 3A = 9 \Rightarrow A = 3 \\ B = -1 \end{cases}$$

So $$\frac{2x+7}{x^2+x-2} = \frac{3}{x-1} + \frac{-1}{x+2}$$

Therefore:

$$\int \frac{2x^3+3x^2-x+5}{x^2+x-2}\,dx = \int \left(2x+1 + \frac{3}{x-1} + \frac{-1}{x+2}\right)dx$$

$$= x^2 + x + 3\ln|x-1| - \ln|x+2| + C$$

2^{nd} example:

$$\frac{-2x^2+2x-3}{x^3-x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

So

$$-2x^2+2x-3 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$$

1^{st} method:

$$A(x^2-1) + B(x^2+x) + C(x^2-x)$$

$$= \underset{\displaystyle -2}{\underset{\shortparallel}{(A+B+C)}}\,x^2 + \underset{\displaystyle 2}{\underset{\shortparallel}{(B-C)}}\,x - \underset{\displaystyle 3}{\underset{\shortparallel}{A}}$$

$$\Leftrightarrow \begin{cases} A = 3 \\ B-C = 2 \\ A+B+C = -2 \end{cases} \Leftrightarrow \begin{cases} A = 3 \\ B-C = 2 \\ B+C = -5 \end{cases} \Leftrightarrow \begin{cases} A = 3 \\ B = -\frac{3}{2} \\ C = -\frac{7}{2} \end{cases}$$

Then $$\frac{-2x^2+2x-3}{x^3-x} = \frac{3}{x} - \frac{\frac{3}{2}}{x-1} - \frac{\frac{7}{2}}{x+1}$$

2^{nd} method:

$\cancel{A(x^2}$ For $x=0$: $-A = -3 \Rightarrow A = 3$

For $x=1$: $2B = -3 \Rightarrow B = -\frac{3}{2}$

For $x=-1$: $2C = -7 \Rightarrow C = -\frac{7}{2}$

$$\int \frac{x^3-2x^2+x-3}{x^3-x}\,dx = \int \left(1 + \frac{3}{x} - \frac{3}{2}\frac{1}{x-1} - \frac{7}{2}\frac{1}{x+1}\right)dx$$

$$= x + 3\ln|x| - \frac{3}{2}\ln|x-1| - \frac{7}{2}\ln|x+1| + C$$

Example 3:

3^{rd} step:

$$\frac{x+1}{x^2-3x+2} = \frac{A}{x-2} + \frac{B}{x-1}$$

$$\Leftrightarrow \quad x+1 = A(x-1) + B(x-2)$$

For $x=1$: $\quad -B = 2 \Leftrightarrow B = -2$

For $x=2$: $\quad A = 3$

$$\frac{x+1}{x^2-3x+2} = \frac{3}{x-2} - \frac{2}{x-1}$$

So

$$\int \frac{x+1}{x^2-3x+2}\,dx = \int\left(\frac{3}{x-2} - \frac{2}{x-1}\right)dx$$

$$= 3\ln|x-2| - 2\ln|x-1| + C$$