Integration by part   التكامل بالتجزيء

We have already seen:

⚠ $\int f(x)g(x)\,dx \neq \int f(x)\,dx \int g(x)\,dx$

⚠ $\int f(x)g(x)\,dx \neq f(x)\int g(x)\,dx$ if $f(x)$ is not constant

we know that

$$\left(f(x)g(x)\right)' = f'(x)g(x) + f(x)g'(x)$$

Then $f'(x)g(x) = \left(f(x)g(x)\right)' - f(x)g'(x)$

Therefore:

$$\int f'(x)g(x)\,dx = \int \left(f(x)g(x)\right)'\,dx - \int f(x)g'(x)\,dx$$

Hence:

$$\boxed{\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx}$$

Integration by part

Let $u = g(x)$ and $v = f(x)$

Then $du = g'(x)dx$ and $dv = f'(x)dx$

$$\int u\,dv = uv - \int v\,du$$

<u>Examples:</u>

① $\int x e^x dx$

Let $\cancel{u = e^x}$. $u = x$ and $dv = e^x dx$

and $du = dx$ and $v = e^x$

<u>Therefore</u>

$$\int x e^x dx = \int u\,dv = uv - \int v\,du = xe^x - \int e^x dx$$

$$= xe^x - e^x + C$$

<u>we check:</u> $(xe^x - e^x)' = e^x + xe^x - e^x$

$$= xe^x$$

② $\int x \cos x\, dx$

Let $u = x$ and $dv = \cos x\, dx$

we have $du = dx$ and $v = \sin x$

Therefore

$$\int x\cos x\,dx = \int u\,dv = uv - \int v\,du$$

$$= x\sin x - \int \sin x\,dx$$

$$= x\sin x + \cos x + C$$

We check: $(x\sin x + \cos x)' = \cancel{\sin x} + x\cos x - \cancel{\sin x}$

③ $\int x^2 e^x\,dx$

Let $u = x^2$ and $dv = e^x\,dx$

We have $du = 2x\,dx$ and $v = e^x$

Therefore $\int x^2 e^x\,dx = \int u\,dv = uv - \int v\,du$

$$= x^2 e^x - \int e^x \times 2x\,dx$$

$\int 2x e^x\,dx = ?$

Let $u = 2x$ and $dv = e^x\,dx$

We have $du = 2\,dx$ and $v = e^x$

Therefore $\int 2x e^x\,dx = \int u\,dv = uv - \int v\,du$

$$= 2xe^x - \int 2e^x\,dx$$

$$= 2xe^x - 2e^x + C$$

Hence:

$$\int x^2 e^x dx = x^2 e^x - (2x e^x - 2e^x) + C$$
$$= x^2 e^x - 2xe^x + 2e^x + C$$

We check:

$$(x^2 e^x - 2xe^x + 2e^x)' = \cancel{2xe^x} + x^2 e^x \cancel{- 2e^x} \cancel{- 2xe^x} \cancel{+ 2e^x}$$
$$= x^2 e^x$$

(4) $\int \ln x \, dx$ $\qquad (\ln x)' = \frac{1}{x}$

Let $u = \ln x$ and $dv = dx$

we have $du = \frac{1}{x} dx$ and $v = x$

Therefore:

$$\int \ln x \, dx = \int u \, dv = uv - \int v \, du = x \ln x - \int x \frac{1}{x} dx$$
$$= x \ln x - \int x \frac{1}{\cancel{x}} dx = x \ln x - x + C$$

We check: $(x \ln x - x)' = \ln x + \cancel{x} \frac{1}{\cancel{x}} - \cancel{1}$
$= \ln x$

(5) $\int \sin^{-1} x \, dx$ $\qquad (\sin^{-1} x)' = \frac{1}{\sqrt{1 - x^2}}$

Let $u = \sin^{-1} x$ and $dv = dx$

We have $du = \frac{1}{\sqrt{1-x^2}} dx$ and $v = x$

Therefore $\int \sin^{-1} x \, dx = \int u \, dv = uv - \int v \, du$

$$= x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx$$

Let $u = 1 - x^2$. We have $du = -2x \, dx$

and $x \, dx = -\frac{1}{2} du$

Therefore $\int \frac{x}{\sqrt{1-x^2}} dx = -\frac{1}{2} \int u^{-1/2} du = -\frac{1}{2} \frac{u^{1/2}}{1/2} + C$

$$= -\sqrt{1-x^2} + C$$

Hence $\int \sin^{-1} x \, dx = x \sin^{-1} x + \sqrt{1-x^2} + C$

<u>We check</u>: $\left(x \sin^{-1} x + \sqrt{1-x^2}\right)' = \sin^{-1} x + \cancel{\frac{x}{\sqrt{1-x^2}}}$

$$+ \frac{1}{\cancel{2}} \cancel{\frac{-2x}{\sqrt{1-x^2}}}$$

(6) $\int \tan^{-1} x \, dx$ $\qquad$ $\tan^{-1} x = \frac{1}{1+x^2}$

Let $u = \tan^{-1} x$ and $dv = dx$

We have $du = \frac{1}{1+x^2} dx$ and $v = x$

We deduce:

$$\int \tan^{-1} x \, dx = \int u \, dv = uv - \int v \, du$$
$$= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} dx$$
$$= x \tan^{-1} x - \frac{1}{2} \ln |1+x^2| + C$$
$$= x \tan^{-1} x - \frac{1}{2} \ln (1+x^2) + C$$

Exercise:

Compute $\int x^2 \sin x \, dx$