Integrals and primitives

If $f(x) = x^2 + \sin x - \frac{1}{x}$ find its derivative:

Answer $f'(x) = 2x + \cos x - (-1 \cdot x^{-2})$

$= 2x + \cos x + \frac{1}{x^2}$.

If $f'(x) = 2x + \cos x + \frac{1}{x^2}$. Give an example of such $f(x)$.

Answer: $f(x) = x^2 + \sin x - \frac{1}{x}$.

$f(x)$ is called an antiderivative or a primitive of $f'(x)$.

Definition: An antiderivative / primitive of a function $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$.

Examples:

$f(x)$	$F(x)$
1	x
x	$\frac{x^2}{2}$
x^2	$\frac{x^3}{3}$

$f(x)$	$F(x)$
x^n	$\frac{x^{n+1}}{n+1}$ $(n \neq -1)$
x^r	$\frac{x^{r+1}}{r+1}$; $r \in \mathbb{R}$, $r \neq -1$
$\sqrt{x} = x^{1/2}$	$\frac{2x^{3/2}}{3}$
$\frac{1}{x}$	$\ln\lvert x \rvert$
$\frac{1}{x^2} = x^{-2}$	$\frac{x^{-1}}{-1} = -\frac{1}{x}$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$

Example:

If $F_1(x) = x + 2$
and $F_2(x) = x + 4$
then $F_1'(x) = 1 = F_2'(x)$
So $F_1'(x) = F_2'(x)$

In general, if $F(x)$ is an anti-derivative of $f(x)$ then $F(x) + C$ is also an anti-derivative of $f(x)$

Theorem: If $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ on an interval then $F(x) - G(x)$ is constant

We denote this:

$$\int f(x)\,dx = F(x) + C$$

Examples:

$$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C \qquad r \neq -1$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\sec^2 x = \frac{1}{\cos^2 x} = 1 + \tan^2 x$$

$$\int \csc^2 x\,dx = -\cot x + C$$

$$\csc^2 x = \frac{1}{\sin^2 x} = 1 + \cot^2 x$$

$$\int \sec x \tan x\,dx = \sec x + C$$

$$\int \csc x \cot x\,dx = -\csc x + C$$

$$\int e^x\,dx = e^x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$$

$$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}x + C$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C$$

Properties:

① $\left(f(x) \pm g(x)\right)' = f'(x) \pm g'(x)$

then $\int \left(f(x) + g(x)\right)dx = \int f(x)\,dx + \int g(x)\,dx$

② $\left(k f(x)\right)' = k f'(x)$

then $\int k f(x)\,dx = k \int f(x)\,dx$

Example:

$$\int \left(2x^2 + 3x - \frac{4}{x} + 2\cos x\right)dx$$

$$= \int 2x^2\,dx + \int 3x\,dx - \int \frac{4}{x}\,dx + \int 2\cos x\,dx$$

$$= 2\int x^2\,dx + 3\int x\,dx - 4\int \frac{1}{x}\,dx + 2\int \cos x\,dx$$

$$= 2\frac{x^3}{3} + 3\frac{x^2}{2} - 4\ln|x| + 2\sin x + C$$

(3) $(f(x)g(x))' \neq f'(x)g'(x)$

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

$$(x\sin x)' = \sin x + x\cos x$$

Then $\cancel{\int f(x)g(x)\,dx \neq \int f(x)\,dx \int g(x)\,dx}$

Zero

$$\cancel{\int f(x)g(x)\,dx \neq f(x)\int g(x)\,dx}$$

Zero

$$\cancel{\int x\sin x\,dx = \int x\,dx \int \sin x\,dx = \frac{x^2}{2}\cdot(-\cos x) + C}$$

Zero

$$\cancel{\int x\sin x\,dx = x\int \sin x\,dx = x(-\cos x) + C}$$

Zero

(4) $$\cancel{\int \frac{f(x)}{g(x)}\,dx = \frac{\int f(x)\,dx}{\int g(x)\,dx}}$$

Zero

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{g(x)^2}$$

$$\neq$$

$$\cancel{\frac{f'(x)}{g'(x)}}$$

Zero

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(2) Method of integration:

(1) By substitution

(2) By part

(3) Rational functions