Example: Find the elements of the hyperbola of equation

$$4x^2 - y^2 + 8x + 4y - 36 = 0$$

and sketch it.

Solution: Completing the square:

$$\begin{aligned}
4x^2 + 8x &= 4(x^2 + 2x) \\
&= 4(x^2 + 2\times 1\times x + 1^2 - 1^2) \\
&= 4\left[(x+1)^2 - 1\right] \\
&= 4(x+1)^2 - 4.
\end{aligned}$$

$$\begin{aligned}
-y^2 + 4y &= -1(y^2 - 4y) \\
&= -1(y^2 - 2\times 2y + 2^2 - 2^2) \\
&= -1((y-2)^2 - 4) \\
&= -(y-2)^2 + 4
\end{aligned}$$

The equation becomes:

$$4(x+1)^2 - 4 - (y-2)^2 + 4 - 36 = 0$$

$$\Leftrightarrow 4(x+1)^2 - (y-2)^2 = 36$$

$$\Leftrightarrow \quad \frac{(x+1)^2}{3^2} - \frac{(y-2)^2}{6^2} = 1$$

$a = 3$; $\cancel{g}$ $b = 6$; $c = \sqrt{9+36} = \sqrt{45}$

$= 3\sqrt{5}$

$36 < 45 < 49$

$6 < \sqrt{45} < 7$

The bisector line of the hyperbola is horizontal

The elements:

① The center $C(-1, 2)$

② The bisector line : $y = 2$

③ The vertices : $V_1(2, 2)$

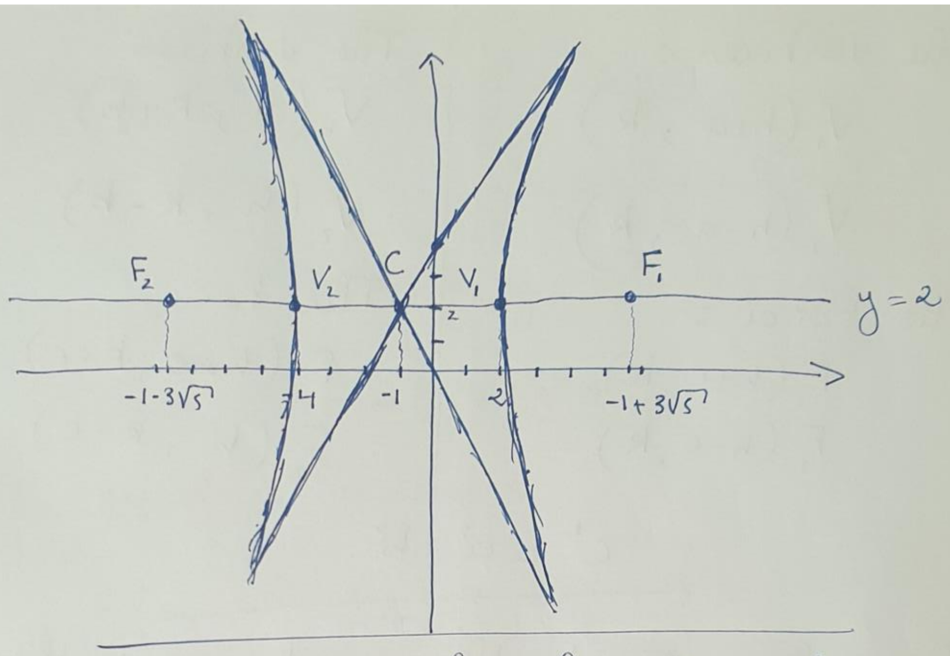
$V_2(-4, 2)$

④ The focci : $F_1(-1+\sqrt{45}, 2)$

$F_2(-1-\sqrt{45}, 2)$

⑤ The asymptotes :

$$y - 2 = \pm \underset{\substack{\uparrow \\ 6/3}}{2}(x+1)$$



Example: Find the elements of the hyperbola of equation

$$5x^2 - y^2 + 40x - 6y + 76 = 0$$

and sketch it.

Solution: ① Completing the square:

$$5x^2 + 40x = 5\left(x^2 + 8x\right)$$

$$= 5\left(x^2 + 2\times 4x + 4^2 - 4^2\right)$$

$$= 5\left[(x+4)^2 - 16\right]$$

$$= 5(x+4)^2 - 80$$

$$-y^2 - 6y = -1\left(y^2 + 6y\right)$$
$$= -1\left(y^2 + 2\times 3y + 3^2 - 3^2\right)$$
$$= -1\left[(y+3)^2 - 9\right]$$
$$= -(y+3)^2 + 9$$

The equation becomes:

$$5(x+4)^2 - 80 - (y+3)^2 + 9 + 76 = 0$$
$$\Leftrightarrow 5(x+4)^2 - (y+3)^2 + 5 = 0$$

$$\Leftrightarrow 5 = (y+3)^2 - 5(x+4)^2$$
$$\Leftrightarrow \boxed{\frac{(y+3)^2}{(\sqrt{5})^2} - \frac{(x+4)^2}{1^2} = 1}$$

$$\Leftrightarrow 5(x+4)^2 - (y+3)^2 = -5$$
$$\Leftrightarrow -\frac{(x+4)^2}{1^2} + \frac{(y+3)^2}{(\sqrt{5})^2} = 1$$

The elements:

1. The center $C(-4, -3)$
2. The bisector line : $x = -4$
3. The vertices : $V_1(-4; -3+\sqrt{5})$; $V_2(-4; -3-\sqrt{5})$

④ The focci: $c = \sqrt{1+5} = \sqrt{6}$

$F_1(-4, -3+\sqrt{6})$; $F_2(-4, -3-\sqrt{6})$

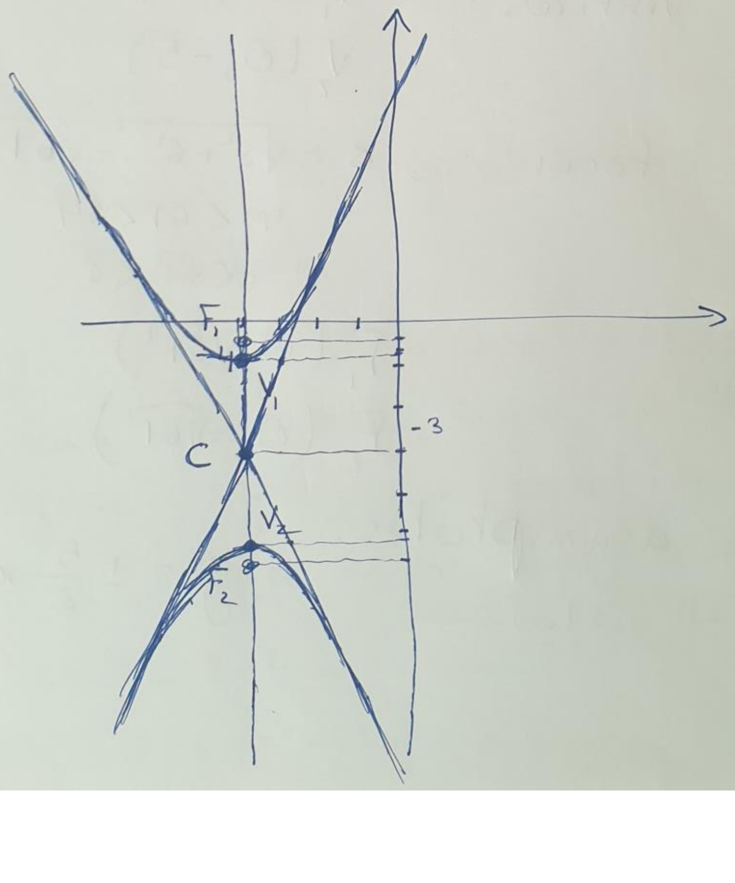
⑤ The asymptotes:

$$y+3 = \pm \frac{\sqrt{5}}{1}(x+4)$$

$$4 < 5 < 9$$

$$\Rightarrow 2 < \sqrt{5} < 3$$

We sketch:



In general: If we have an equation of the form:

$$ax^2 + by^2 + cx + dy + e = 0$$

with $(a,b) \neq (0,0)$

a,b معا لا يساويان صفر

لكن يمكن لأحدهما أن يساوي صفراً

We have three cases:

① One of a or b is equal to zero

Example: $2x^2 + 3x - y + 1 = 0$

This is the equation of a parabola.

② If a and b are $\neq 0$ and have the same sign. In this case it is an equation of an ellipse (or empty set)

$$-x^2 - 3y^2 + \cdots = 0$$

$$\Leftrightarrow \quad 0 = x^2 + 3y^2 + \cdots$$

(3) If a and b are $\neq 0$ and of different sign. In this case it is the equation of a hyperbola

Example: Give the nature of the conic section of equation

$$4y^2 = 9x^2 + 18x + 45.$$

Solution: The equation is equivalent to : $9x^2 - 4y^2 + 18x + 45 = 0$.

This is the equation of a hyperbola.