Completing the square إكمال المربع

Recall that: $(a+b)^2 = a^2 + 2ab + b^2$

$(a-b)^2 = a^2 - 2ab + b^2$

Examples: $(x+3)^2 = x^2 + 6x + 9$

$(x-1)^2 = x^2 - 2x + 1$

$(x-4)^2 = x^2 - 8x + 16$

$(x+5)^2 = x^2 + 10x + 25$

Completing the square:

Examples:

$$x^2 - 4x = x^2 - 2\times 2x + 2^2 - 2^2$$
$$= (x-2)^2 - 4 \quad \checkmark$$

$$x^2 + 6x = x^2 + 2\times 3x + 3^2 - 3^2$$
$$= (x+3)^2 - 9$$

$$x^2 + x = x^2 + 2\frac{1}{2}x + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2$$
$$= \left(x + \frac{1}{2}\right)^2 - \frac{1}{4}$$

$$x^2 - 3x = x^2 - 2\times\frac{3}{2}x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2$$
$$= \left(x - \frac{3}{2}\right)^2 - \frac{9}{4}$$

Problem: Find the elements of the parabola of equation $2x^2 + 8x - 3y + 5 = 0$ and sketch it.

Solution:

We have:

$$2x^2 + 8x = 2\left[x^2 + 4x\right]$$

$$= 2\left[x^2 + 2 \times 2x + 2^2 - 2^2\right]$$

$$= 2\left[(x+2)^2 - 4\right]$$

$$= 2(x+2)^2 - 8$$

The equation becomes:

$$2(x+2)^2 - 8 - 3y + 5 = 0$$

$$\Leftrightarrow 2(x+2)^2 = 3y + 3$$

$$\Leftrightarrow 2(x+2)^2 = 3(y+1)$$

$$\Leftrightarrow (x+2)^2 = \frac{3}{2}(y+1)$$

$$\Leftrightarrow \boxed{(x+2)^2 = 4\,\frac{3}{8}(y+1)}$$

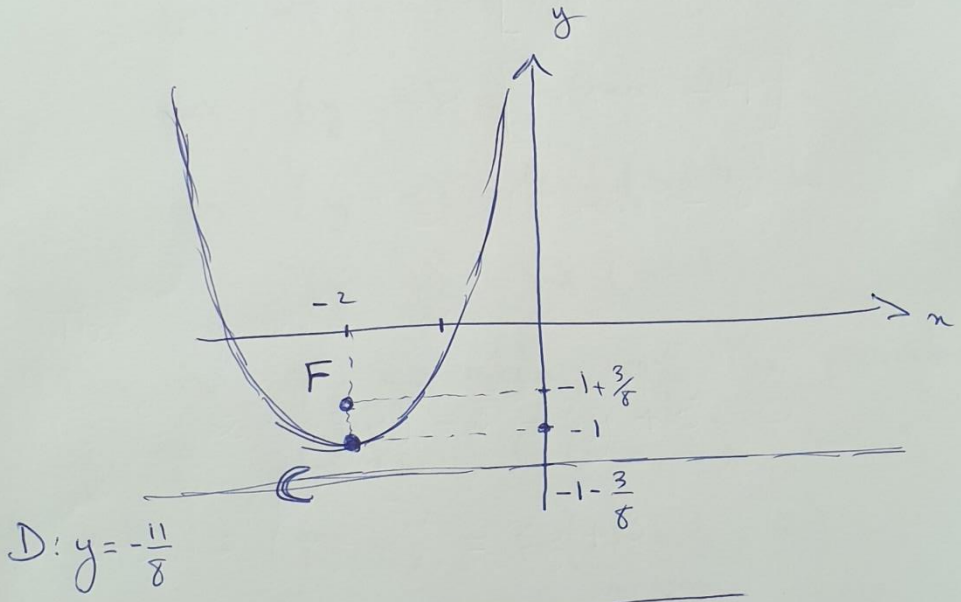
$$(x-h)^2 = 4a(y-k)$$

This is an equation of the parabola of :

Center $C(-2;-1)$

Focus $F\left(-2;\underbrace{-1+\frac{3}{8}}_{-\frac{5}{8}}\right)$

Directrix $D: y=\underbrace{-1-\frac{3}{8}}_{-\frac{11}{8}}$



Example: The same question for the equation: $2y^2-8y-4x-8=0$

Solution: We have

$$2y^2-8y=2\left[y^2-4y\right]=2\left[y^2-2\times 2y+2^2-2^2\right]$$

$$= 2\left[(y-2)^2 - 4\right]$$

$$= 2(y-2)^2 - 8$$

Therefore, the equation is equivalent to:

$$2(y-2)^2 - 8 - 4x - 8 = 0$$

$$\Leftrightarrow 2(y-2)^2 = 4x + 16$$

$$\Leftrightarrow 2(y-2)^2 = 4(x+4)$$

$$\Leftrightarrow (y-2)^2 = 2(x+4)$$

$$\Leftrightarrow (y-2)^2 = 4 \times \frac{1}{2}(x+4)$$

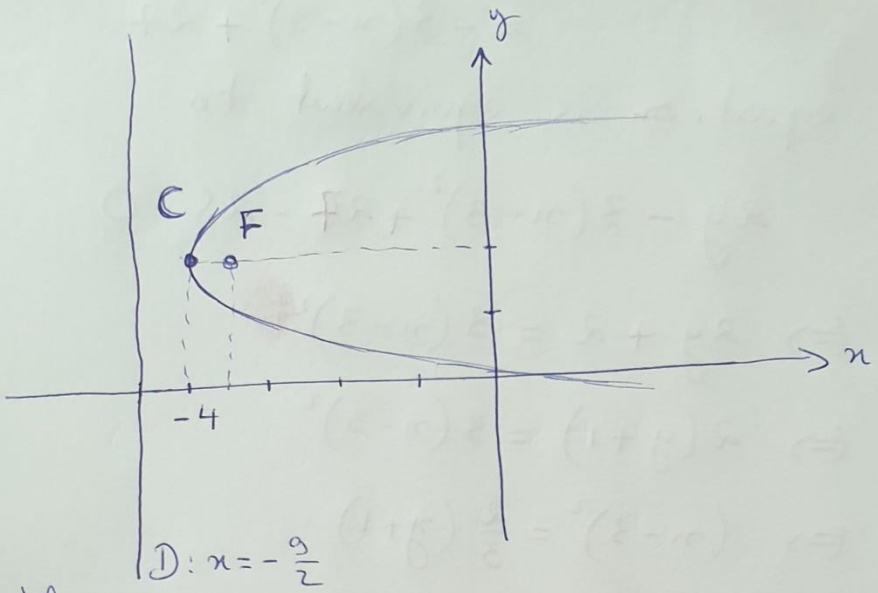
$$(y-k)^2 = 4a(x-h).$$

This is an equation of the parabola of:

Center $C(-4, 2)$

Focus $F(\underbrace{-4+\frac{1}{2}}_{-\frac{7}{2}}, 2)$

Directrix $D: x = \underbrace{-4-\frac{1}{2}}_{-\frac{9}{2}}$



Problem:
The same question for $x^2+2y+2x=2$.

Solution: we have $x^2+2x = x^2+2\times1\times x+1^2-1^2$
$= (x^2+1)^2-1$

Therefore, the equation is equivalent to:

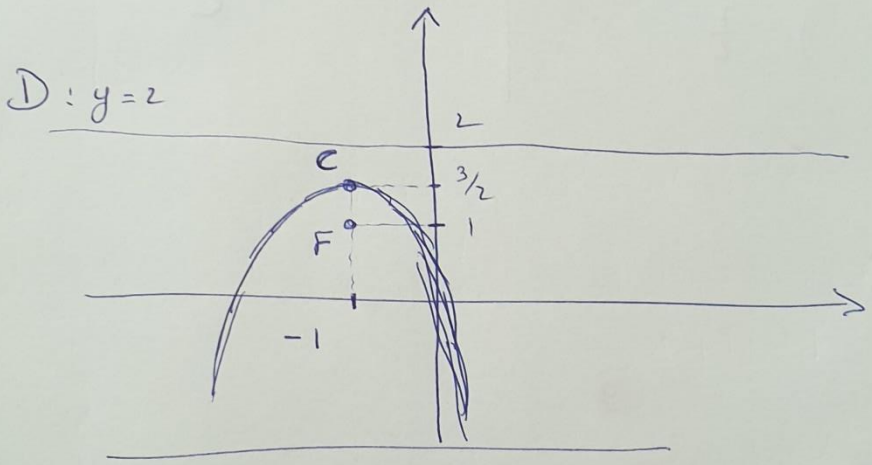
$$(x+1)^2-1+2y=2$$
$$\Leftrightarrow (x+1)^2 = -2y+3$$
$$\Leftrightarrow (x+1)^2 = -2\left(y-\frac{3}{2}\right)$$
$$\Leftrightarrow (x+1)^2 = -4\times\frac{1}{2}\left(y-\frac{3}{2}\right)$$
$$(x-h)^2 = 4a\,(y-k)$$
$$a=-\frac{1}{2}$$

This is an equation of the parabola of :

Center $C\left(-1 ; \frac{3}{2}\right)$

Focus $F(-1 ; 1)$

Directrix $D : y = 2$.



Example: The same question for :

$$2y = 3x^2 + 18x - 25 = 0$$

Solution: we have

$$-3x^2 + 18x = -3\left[x^2 - 6x\right]$$

$$= -3\left[x^2 - 2 \times 3x + 3^2 - 3^2\right]$$

$$= -3\left[(x-3)^2 - 9\right]$$

$$= -3(x-3)^2 + 27$$

The equation is equivalent to:

$$2y - 3(x-3)^2 + 27 - 25 = 0$$

$$\Leftrightarrow 2y + 2 = 3(x-3)^2$$

$$\Leftrightarrow 2(y+1) = 3(x-3)^2$$

$$\Leftrightarrow (x-3)^2 = \frac{2}{3}(y+1)$$

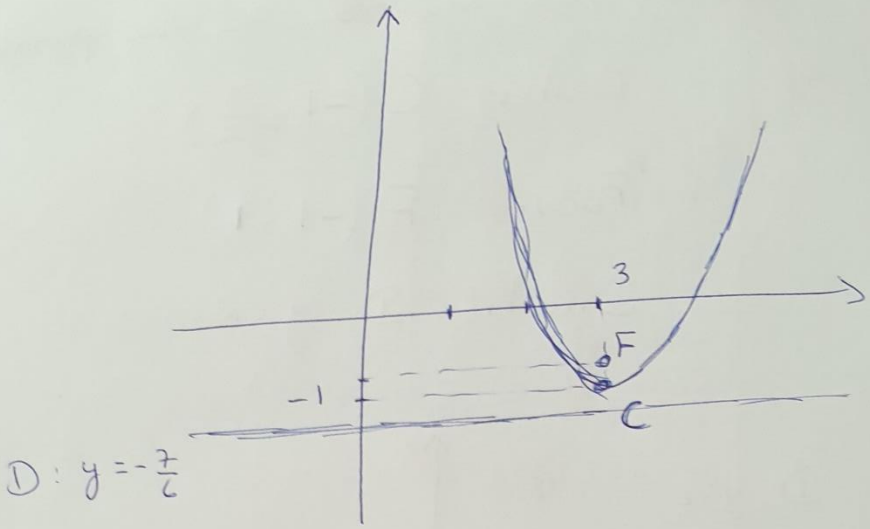
$$\Leftrightarrow (x-3)^2 = 4 \times \underbrace{\frac{2}{12}}_{\frac{1}{6}}(y+1)$$

$$(x-h)^2 = 4a(y-k).$$

This is an equation of a parabola of: Center $C(3; -1)$

Focus $F\left(3; \underbrace{-1+\frac{1}{6}}_{-\frac{5}{6}}\right)$

Directrix $D: y = \underbrace{-1-\frac{1}{6}}_{-\frac{7}{6}}$.



D : $y = -\frac{7}{6}$