Research Article

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Asymptotic study of Leray solution of 3D-Navier-Stokes equations with exponential damping

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Abstract: We study the uniqueness, the continuity in L^2, and the large time decay for the Leray solutions of the 3D incompressible Navier-Stokes equations with the nonlinear exponential damping term $a(e^{b|u|^2} - 1)u$, $(a, b > 0)$.

Keywords: Navier-Stokes equations, Friedrich method, global weak solution

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1 Introduction

In this article, we investigate the questions of the existence, uniqueness, and asymptotic study of global weak solution to the following modified incompressible Navier-Stokes equations in $\mathbb{R}^3$:

$$\begin{cases} \partial_t u - \nu\Delta u + u \cdot \nabla u + a\left(e^{b|u|^2} - 1\right)u = -\nabla p & \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ u(0, x) = u^0(x) & \text{in } \mathbb{R}^3, \\ a, b > 0, \end{cases} \qquad (S)$$

where $u = u(t, x) = (u_1, u_2, u_3)$ and $p = p(t, x)$ denote, respectively, the unknown velocity and the unknown pressure of the fluid at the point $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^3$. The function ν denotes the viscosity of the fluid and $u^0 = (u_1^0(x), u_2^0(x), u_3^0(x))$ is the initial given velocity. The damping of the system comes from the resistance to the motion of the flow. It describes various physical situations, such as porous media flow, drag or friction effects, and some dissipative mechanisms (see [1–4] and references therein). The fact that $\operatorname{div} u = 0$ allows us to write the term $(u \cdot \nabla u) := u_1\partial_1 u + u_2\partial_2 u + u_3\partial_3 u$ in the following form: $\operatorname{div}(u \otimes u) := (\operatorname{div}(u_1 u), \operatorname{div}(u_2 u), \operatorname{div}(u_3 u))$. If the initial velocity u^0 is quite regular, the divergence-free condition determines uniquely the pressure p.

Without the loss of generality and in order to simplify the proofs of our results, we consider the viscosity unitary ($\nu = 1$).

The global existence of weak solution to the initial value problem of classical incompressible Navier-Stokes was proved by Leray and Hopf (see [5,6]) long before. Uniqueness remains an open problem if the dimension $d \geq 3$.

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Polynomial damping $\alpha|u|^{\beta-1}u$ is studied in [7] by Cai and Jiu. They proved the global existence of weak solution in

$$L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3)) \cap L^{\beta+1}(\mathbb{R}^+, L^{\beta+1}(\mathbb{R}^3)).$$

The exponential damping $a(e^{b|u|^2}-1)u$ is studied in [8] by Benameur. He proved the existence of a global weak solution in

$$L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3)) \cap \mathcal{E}_b,$$

where $\mathcal{E}_b = \{f : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R} : \text{measurable}, \ (e^{b|f|^2}-1)|f|^2 \in L^1(\mathbb{R}^+ \times \mathbb{R}^3)\}$.

The purpose of this article is to prove the uniqueness and the continuity of the global solution given in [8]. Using the Friedrich method, we construct approximate solutions and make more delicate estimates to proceed with the compactness arguments. In particular, we obtain some new *a priori* estimates as follows:

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(s)\|_{L^2}^2 ds + 2a\int_0^t \|\left(e^{b|u(s)|^2}-1\right)|u(s)|^2\|_{L^1} ds \le \|u^0\|_{L^2}^2.$$

To prove its uniqueness, we use the energy method and approximate systems. The proof of the asymptotic study is based on a decomposition of the solution in high and low frequencies and the uniqueness of such solution in a well-chosen time t_0.

In our case of exponential damping, we find more regularity of Leray solution in $\cap_p L^p(\mathbb{R}^+, L^p(\mathbb{R}^3))$. In particular, we give a new energy estimate. Our main result is the following:

Theorem 1.1. *Let $u^0 \in L^2(\mathbb{R}^3)$ be a divergence-free vector fields, then there is a unique global solution of the system (S): $u \in C_b(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3)) \cap \mathcal{E}_b$. Moreover, for all $t \ge 0$,*

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(s)\|_{L^2}^2 ds + 2a\int_0^t \|\left(e^{b|u(s)|^2}-1\right)|u(s)|^2\|_{L^1} ds \le \|u^0\|_{L^2}^2. \tag{1.1}$$

Moreover, we have

$$\limsup_{t\to\infty} \|u(t)\|_{L^2} = 0. \tag{1.2}$$

Remark 1.2.
(1) The new results in this theorem are the uniqueness, the continuity of the global weak solution in $L^2(\mathbb{R}^3)$, and its asymptotic behavior at infinity.
(2) The uniqueness of weak solution implies that

$$\|u(t_2)\|_{L^2}^2 + 2\int_{t_1}^{t_2} \|\nabla u(s)\|_{L^2}^2 ds + 2a\int_{t_1}^{t_2} \|\left(e^{b|u(s)|^2}-1\right)|u(s)|^2\|_{L^1} ds \le \|u(t_1)\|_{L^2}^2, \tag{1.3}$$

which implies that $(t \to \|u(t)\|_{L^2})$ is decreasing.

2 Notations and preliminary results

The Friedrich operator J_R is defined for $R > 0$ by: $J_R(D)f = \mathcal{F}^{-1}(\chi_{B_R}\widehat{f})$, where B_R is the ball of center 0 and radius R and $f \in L^2(\mathbb{R}^3)$. The Leray operator $\mathbb{P}$ is the projector operator of $(L^2(\mathbb{R}^3))^3$ on the space of divergence-free vector fields $L^2_\sigma(\mathbb{R}^3)$.

If f is in the Schwartz space $(\mathcal{S}(\mathbb{R}^n))^3$,

$$\mathcal{F}(\mathbb{P}f) = \widehat{f}(\xi) - \left(\widehat{f}(\xi).\ \frac{\xi}{|\xi|}\right)\frac{\xi}{|\xi|} = M(\xi)\widehat{f}(\xi)$$

and $(\mathbb{P}f)_k(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}\left(\delta_{kj} - \frac{\xi_k\xi_j}{|\xi|^2}\right)\widehat{f_j}(\xi)e^{i\xi\cdot x}d\xi$, where $M(\xi)$ is the matrix $(\delta_{k,\ell} - \frac{\xi_k\xi_\ell}{|\xi|^2})_{1\le k,\ell\le 3}$.

Define also the operator $A_R(D)$ on $L^2(\mathbb{R}^3)$ by:

$$A_R(D)u = \mathbb{P}J_R(D)u = \mathcal{F}^{-1}\Big(M(\xi)\chi_{B_R}(\xi)\hat{u}\Big).$$

To simplify the exposition of the main result, we first collect some preliminary results and give some new technical lemmas.

Proposition 2.1. [9] *Let H be a Hilbert space.*

(1) *The unit ball is weakly compact, i.e., if (x_n) is a bounded sequence in H, then there is a subsequence $(x_{\varphi(n)})$ such that*

$$(x_{\varphi(n)}|y) \to (x|y), \quad \forall y \in H.$$

(2) *If $x \in H$ and (x_n) is a bounded sequence in H such that $\lim_{n\to+\infty}(x_n|y) = (x|y)$, for all $y \in H$, then $\|x\| \le \liminf_{n\to\infty}\|x_n\|$.*

(3) *If $x \in H$ and (x_n) is a bounded sequence in H such that $\lim_{n\to+\infty}(x_n|y) = (x|y)$,* for all *$y \in H$ and $\limsup_{n\to\infty}\|x_n\| \le \|x\|$, then $\lim_{n\to\infty}\|x_n - x\| = 0$.*

We recall the following product law in the homogeneous Sobolev spaces:

Lemma 2.2. [10] *Let s_1 and s_2 be two real numbers and $d \in \mathbb{N}$.*

(1) *If $s_1 < \frac{d}{2}$ and $s_1 + s_2 > 0$, there exists a constant $C_1 = C_1(d, s_1, s_2)$ such that: if $f, g \in \dot{H}^{s_1}(\mathbb{R}^d) \cap \dot{H}^{s_2}(\mathbb{R}^d)$, then $f.g \in \dot{H}^{s_1+s_2-\frac{d}{2}}(\mathbb{R}^d)$ and*

$$\|fg\|_{\dot{H}^{s_1+s_2-\frac{d}{2}}} \le C_1\Big(\|f\|_{\dot{H}^{s_1}}\|g\|_{\dot{H}^{s_2}} + \|f\|_{\dot{H}^{s_2}}\|g\|_{\dot{H}^{s_1}}\Big).$$

(2) *If $s_1, s_2 < \frac{d}{2}$ and $s_1 + s_2 > 0$, there exists a constant $C_2 = C_2(d, s_1, s_2)$ such that: if $f \in \dot{H}^{s_1}(\mathbb{R}^d)$ and $g \in \dot{H}^{s_2}(\mathbb{R}^d)$, then $f.g \in \dot{H}^{s_1+s_2-\frac{d}{2}}(\mathbb{R}^d)$ and*

$$\|fg\|_{\dot{H}^{s_1+s_2-\frac{d}{2}}} \le C_2\|f\|_{\dot{H}^{s_1}}\|g\|_{\dot{H}^{s_2}}.$$

Lemma 2.3. [11] *Let $\beta > 0$ and $d \in \mathbb{N}$. Then, for all $x, y \in \mathbb{R}^d$, we have*

$$\langle |x|^\beta x - |y|^\beta y, x - y\rangle \ge \frac{1}{2}(|x|^\beta + |y|^\beta)|x - y|^2, \tag{2.1}$$

and

$$\langle \big(e^{b|x|^2} - 1\big)x - \big(e^{b|y|^2} - 1\big)y, x - y\rangle \ge \frac{1}{2}(\big(e^{b|x|^2} - 1\big) + \big(e^{b|y|^2} - 1\big))|x - y|^2. \tag{2.2}$$

Proposition 2.4. [8] *Let $\nu_1, \nu_2, \nu_3 \in [0, \infty)$, $r_1, r_2, r_3 \in (0, \infty)$, and $f^0 \in L^2_\sigma(\mathbb{R}^3)$.*

For $n \in \mathbb{N}$, let $F_n : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}^3$ be a measurable function in $C^1(\mathbb{R}^+, L^2(\mathbb{R}^3))$ such that

$$A_n(D)F_n = F_n,\ \ F_n(0, x) = A_n(D)f^0(x)$$

and

$(E1)$ $\partial_t F_n + \sum_{k=1}^3 \nu_k|D_k|^{2r_k}F_n + A_n(D)\operatorname{div}(F_n \otimes F_n) + A_n(D)h(|F_n|)F_n = 0.$

$(E2)$

$$\|F_n(t,\cdot)\|^2_{L^2} + 2\sum_{k=1}^3 \nu_k\int_0^t \||D_k|^{r_k}F_n(s,\cdot)\|^2_{L^2}ds + 2a\int_0^t \|h(|F_n(s,\cdot)|)|F_n(s,\cdot)|^2\|_{L^1}ds \le \|f^0\|^2_{L^2},$$

where $h(\lambda) = a(e^{b\lambda^2} - 1)$, *with* $a, b > 0$. *Then, for every* $\varepsilon > 0$, *there is* $\delta = \delta(\varepsilon, a, b, \nu_1, \nu_2, \nu_3, r_1, r_2, r_3, \|f^0\|_{L^2}) > 0$ *such that for all* $t_1, t_2 \in \mathbb{R}^+$, *we have*

$$(|t_2 - t_1| < \delta \;\Rightarrow\; \|F_n(t_2) - F_n(t_1)\|_{H^{-s_0}} < \varepsilon), \;\; \forall n \in \mathbb{N}, \tag{2.3}$$

with $s_0 \geq \max(3, 2r_1, 2r_2, 2r_3)$.

Lemma 2.5. *Let* $a, b > 0$, *then there is a unique real* $\lambda_0 = \lambda_0(a, b) \geq 0$ *such that for all* $\lambda \geq 0$,

$$a(e^{b\lambda} - 1) \leq \lambda \;\Rightarrow\; \lambda \in [0, \lambda_0].$$

Precisely,
- *if* $ab \geq 1$, *we have* $\lambda_0 = 0$,
- *if* $ab < 1$, *we have* $\lambda_0 > \frac{1}{b}\log(1/ab)$.

3 Proof of the main Theorem 1.1

The proof of the theorem is given in four steps:

3.1 Existence of weak solution

In this step, we build approximate solutions of the system (S) inspired by the method used in [8,10], hence we construct a global solution. For this, consider the approximate system with parameter $n \in \mathbb{N}$ as follows:

$$(S_n)\begin{cases} \partial_t u - \Delta J_n u + J_n(J_n u \cdot \nabla J_n u) + aJ_n\Big[\Big(e^{b|J_n u|^2} - 1\Big)J_n u\Big] = -\nabla p_n \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ p_n = (-\Delta)^{-1}(\operatorname{div} J_n(J_n u \cdot \nabla J_n u) + a\operatorname{div} J_n\Big[\Big(e^{b|J_n u|^2} - 1\Big)J_n u\Big]) \\ \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ u(0, x) = J_n u^0(x) \quad \text{in } \mathbb{R}^3. \end{cases}$$

J_n is the Friedrich operator defined in the section 2.
- By Cauchy-Lipschitz theorem, we obtain a unique solution $u_n \in C^1(\mathbb{R}^+, L^2_\sigma(\mathbb{R}^3))$ of $(S_{2,n})$. Moreover, $J_n u_n = u_n$ such that

$$\|u_n(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u_n\|_{L^2}^2 + 2a\int_0^t \|\Big(e^{b|u_n|^2} - 1\Big)|u_n|^2\|_{L^1} \leq \|u^0\|_{L^2}^2. \tag{3.1}$$

- The sequence $(u_n)_n$ is bounded on $L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3))$ and $L^2(\mathbb{R}^+, \dot H^1(\mathbb{R}^3))$. Using Proposition 2.4 and the interpolation method, we deduce that the sequence $(u_n)_n$ is equicontinuous on $H^{-1}(\mathbb{R}^3)$.
- Let $(T_q)_q$ be a strictly increasing sequence such that $\lim_{q\to+\infty} T_q = \infty$. Consider a sequence of functions $(\theta_q)_q$ in $C_0^\infty(\mathbb{R}^3)$ such that

$$\begin{cases} \theta_q(x) = 1, & \text{for } |x| \leq q + \frac{5}{4} \\ \theta_q(x) = 0, & \text{for } |x| \geq q + 2 \\ 0 \leq \theta_q \leq 1. \end{cases}$$

Using (3.1), the equicontinuity of the sequence $(u_n)_n$ on $H^{-1}(\mathbb{R}^3)$, and the classical argument by combining Ascoli's theorem and the Cantor diagonal process, there exists a subsequence $(u_{\varphi(n)})_n$ and $u \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap C(\mathbb{R}^+, H^{-3}(\mathbb{R}^3))$ such that for all $q \in \mathbb{N}$,

$$\lim_{n\to\infty} \|\theta_q(u_{\varphi(n)} - u)\|_{L^\infty([0,T_q],H^{-4})} = 0. \tag{3.2}$$

In particular, the sequence $(u_{\varphi(n)}(t))_n$ converges weakly in $L^2(\mathbb{R}^3)$ to $u(t)$ for all $t \geq 0$.

- Combining the above inequalities, we obtain

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(s)\|_{L^2}^2 ds + 2a\int_0^t \|\left(e^{b|u(s)|^2} - 1\right)|u(s)|^2\|_{L^1} ds \leq \|u^0\|_{L^2}^2, \tag{3.3}$$

for all $t \geq 0$.

- u is a solution to the system (S).

3.2 Continuity of the solution in L^2

In this section, we give a simple proof of continuity of the solution u of the system (S), and we prove also that $u \in C(\mathbb{R}^+, L^2(\mathbb{R}^3))$. The construction of the solution is based on the Friedrich approximation method. We point out that we can use this method to show the same results as in [4].

- By inequality (3.3), we obtain

$$\limsup_{t\to 0}\|u(t)\|_{L^2} \leq \|u^0\|_{L^2}.$$

Then, proposition 2.1-(3) implies that

$$\limsup_{t\to 0}\|u(t) - u^0\|_{L^2} = 0,$$

which ensures the continuity of u at 0.

- Consider the functions

$$v_{n,\varepsilon}(t,\cdot) = u_{\varphi(n)}(t+\varepsilon,\cdot),\ \ p_{n,\varepsilon}(t,\cdot) = p_{\varphi(n)}(t+\varepsilon,\cdot),$$

for $n \in \mathbb{N}$ and $\varepsilon > 0$. We have

$$\begin{aligned}\partial_t u_{\varphi(n)} - \Delta u_{\varphi(n)} + J_{\varphi(n)}(u_{\varphi(n)}\cdot\nabla u_{\varphi(n)}) + aJ_{\varphi(n)}\left(e^{b|u_{\varphi(n)}|^2} - 1\right)u_{\varphi(n)} &= -\nabla p_{\varphi(n)}\\ \partial_t v_{n,\varepsilon} - \Delta v_{n,\varepsilon} + J_{\varphi(n)}(v_{n,\varepsilon}\cdot\nabla v_{n,\varepsilon}) + aJ_{\varphi(n)}\left(e^{b|v_{n,\varepsilon}|^4} - 1\right)v_{n,\varepsilon} &= -\nabla p_{n,\varepsilon}.\end{aligned}$$

The function $w_{n,\varepsilon} = u_{\varphi(n)} - v_{n,\varepsilon}$ fulfills the following:

$$\begin{aligned}&\partial_t w_{n,\varepsilon} - \Delta w_{n,\varepsilon} + aJ_{\varphi(n)}(\left(e^{b|u_{\varphi(n)}|^2} - 1\right)u_{\varphi(n)} - \left(e^{b|v_{n,\varepsilon}|^2} - 1\right)v_{n,\varepsilon})\\ &\quad = -\nabla(p_{\varphi(n)} - p_{n,\varepsilon}) + J_{\varphi(n)}(w_{n,\varepsilon}\cdot\nabla w_{n,\varepsilon}) - J_{\varphi(n)}(w_{n,\varepsilon}\cdot\nabla u_{\varphi(n)}) - J_{\varphi(n)}(u_{\varphi(n)}\cdot\nabla w_{n,\varepsilon}).\end{aligned}$$

Taking the scalar product in $L^2(\mathbb{R}^3)$ with $w_{n,\varepsilon}$ and using the properties div $w_{n,\varepsilon} = 0$ and $\langle w_{n,\varepsilon}\cdot\nabla w_{n,\varepsilon}, w_{n,\varepsilon}\rangle = 0$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|w_{n,\varepsilon}\|_{L^2}^2 + \|\nabla w_{n,\varepsilon}\|_{L^2}^2 + a\langle J_{\varphi(n)}(\left(e^{b|u_{\varphi(n)}|^2} - 1\right)u_{\varphi(n)} - \left(e^{b|v_{n,\varepsilon}|^4} - 1\right)v_{n,\varepsilon});\ w_{n,\varepsilon}\rangle_{L^2}\\ &\quad = -\langle J_{\varphi(n)}(w_{n,\varepsilon}\cdot\nabla u_{\varphi(n)});\ w_{n,\varepsilon}\rangle_{L^2}.\end{aligned} \tag{3.4}$$

Using inequality (2.2), we obtain

$$\begin{aligned}&\langle J_{\varphi(n)}(\left(e^{b|u_{\varphi(n)}|^2} - 1\right)u_{\varphi(n)} - \left(e^{b|v_{n,\varepsilon}|^2} - 1\right)v_{n,\varepsilon});\ w_{n,\varepsilon}\rangle_{L^2}\\ &= \langle\left(e^{b|u_{\varphi(n)}|^2} - 1\right)u_{\varphi(n)} - \left(e^{b|v_{n,\varepsilon}|^2} - 1\right)v_{n,\varepsilon}; J_{\varphi(n)}w_{n,\varepsilon}\rangle_{L^2}\\ &= \langle\left(e^{b|u_{\varphi(n)}|^2} - 1\right)u_{\varphi(n)} - \left(e^{b|v_{n,\varepsilon}|^2} - 1\right)v_{n,\varepsilon};\ w_{n,\varepsilon}\rangle_{L^2}\\ &\geq \frac{1}{2}\int_{\mathbb{R}^3}(\left(e^{b|u_{\varphi(n)}|^2} - 1\right) + \left(e^{b|v_{n,\varepsilon}|^2} - 1\right))|w_{n,\varepsilon}|^2\\ &\geq \frac{1}{2}\int_{\mathbb{R}^3}\left(e^{b|u_{\varphi(n)}|^2} - 1\right)|w_{n,\varepsilon}|^2,\end{aligned}$$

$$|\langle J_{\varphi(n)}(w_{n,\varepsilon}\cdot\nabla u_{\varphi(n)});\, w_{n,\varepsilon}\rangle_{L^2}| \le \int_{\mathbb{R}^3} |w_{n,\varepsilon}|.\,|u_{\varphi(n)}|.\,|\nabla w_{n,\varepsilon}| \le \frac{1}{2}\int_{\mathbb{R}^3} |w_{n,\varepsilon}|^2|u_{\varphi(n)}|^2 + \frac{1}{2}\|\nabla w_{n,\varepsilon}\|_{L^2}^2.$$

Combining the identity (2.2) and the inequality (3.4), we obtain

$$\frac{d}{dt}\|w_{n,\varepsilon}\|_{L^2}^2 + \|\nabla w_{n,\varepsilon}\|_{L^2}^2 + a\int_{\mathbb{R}^3}\left(e^{b|u_{\varphi(n)}|^2}-1\right)|w_{n,\varepsilon}|^2 \le \int_{\mathbb{R}^3}|w_{n,\varepsilon}|^2|u_{\varphi(n)}|^2.$$

Using Lemma 2.5, we obtain the following:

If $ab \ge 1$

$$\frac{d}{dt}\|w_{n,\varepsilon}\|_{L^2}^2 + \|\nabla w_{n,\varepsilon}\|_{L^2}^2 \le 0.$$

If $ab < 1$, we have

$$\frac{d}{dt}\|w_{n,\varepsilon}\|_{L^2}^2 + \|\nabla w_{n,\varepsilon}\|_{L^2}^2 + a\int_{\mathbb{R}^3}\left(e^{b|u_{\varphi(n)}|^2}-1\right)|w_{n,\varepsilon}|^2 \le +\int_{A_t}|w_{n,\varepsilon}|^2|u_{\varphi(n)}|^2,$$

where $A_{n,t} = \{x\in\mathbb{R}^3/\, a(e^{b|u_{\varphi(n)}(t,x)|^2}-1) < |u_{\varphi(n)}(t,x)|^2\}$. Then, also by Lemma 2.5, we obtain

$$x\in A_{n,t} \;\Rightarrow\; |u_{\varphi(n)}(t,x)|^2 \le \lambda_0,$$

which implies that

$$\frac{d}{dt}\|w_{n,\varepsilon}\|_{L^2}^2 + \|\nabla w_{n,\varepsilon}\|_{L^2}^2 \le \lambda_0\int_{A_t}|w_{n,\varepsilon}|^2 \le \lambda_0\||w_{n,\varepsilon}\|_{L^2}^2.$$

In all cases, we obtain

$$\frac{d}{dt}\|w_{n,\varepsilon}\|_{L^2}^2 \le \lambda_0\||w_{n,\varepsilon}\|_{L^2}^2.$$

By Gronwall lemma, we obtain

$$\|w_{n,\varepsilon}(t)\|_{L^2}^2 \le \|w_{n,\varepsilon}(0)\|_{L^2}^2 e^{\lambda_0 t}.$$

Then,

$$\|u_{\varphi(n)}(t+\varepsilon) - u_{\varphi(n)}(t)\|_{L^2}^2 \le \|u_{\varphi(n)}(\varepsilon) - u_{\varphi(n)}(0)\|_{L^2}^2 e^{\lambda_0 t}.$$

For $t_0 > 0$ and $\varepsilon\in(0,t_0)$, we have

$$\begin{aligned}\|u_{\varphi(n)}(t_0+\varepsilon) - u_{\varphi(n)}(t_0)\|_{L^2}^2 &\le \|u_{\varphi(n)}(\varepsilon) - u_{\varphi(n)}(0)\|_{L^2}^2 \exp(2\lambda_0 t_0),\\ \|u_{\varphi(n)}(t_0-\varepsilon) - u_{\varphi(n)}(t_0)\|_{L^2}^2 &\le \|u_{\varphi(n)}(\varepsilon) - u_{\varphi(n)}(0)\|_{L^2}^2 \exp(2\lambda_0 t_0).\end{aligned}$$

So

$$\begin{aligned}\|u_{\varphi(n)}(\varepsilon) - u_{\varphi(n)}(0)\|_{L^2}^2 &= \|J_{\varphi(n)}u_{\varphi(n)}(\varepsilon) - J_{\varphi(n)}u_{\varphi(n)}(0)\|_{L^2}^2\\ &= \|\chi_{\varphi(n)}(\widehat{u_{\varphi(n)}} - \widehat{u^0})\|_{\varphi(n)}^2\\ &\le \|u_{\varphi(n)}(\varepsilon) - u^0\|_{L^2}^2\\ &\le 2\|u^0\|_{L^2}^2 - 2\mathrm{Re}\,\langle u_{\varphi(n)}(\varepsilon), u^0\rangle.\end{aligned}$$

But $\lim_{n\to+\infty}\langle u_{\varphi(n)}(\varepsilon), u^0\rangle = \langle u(\varepsilon), u^0\rangle$. Hence,

$$\liminf_{n\to\infty}\|u_{\varphi(n)}(\varepsilon) - u_{\varphi(n)}(0)\|_{L^2}^2 \le 2\|u^0\|_{L^2}^2 - 2\mathrm{Re}\,\langle u(\varepsilon);\, u^0\rangle_{L^2}.$$

Moreover, for all $q, N\in\mathbb{N}$,

$$\|J_N(\theta_q.\,(u_{\varphi(n)}(t_0\pm\varepsilon) - u_{\varphi(n)}(t_0)))\|_{L^2}^2 \le \|\theta_q.\,(u_{\varphi(n)}(t_0\pm\varepsilon) - u_{\varphi(n)}(t_0))\|_{L^2}^2 \le \|u_{\varphi(n)}(t_0\pm\varepsilon) - u_{\varphi(n)}(t_0)\|_{L^2}^2.$$

Using (3.2), we obtain, for q big enough,

$$\|J_N(\theta_q.\ (u(t_0 \pm \varepsilon) - u(t_0)))\|_{L^2}^2 \leq \liminf_{n\to\infty}\|u_{\varphi(n)}(t_0 \pm \varepsilon) - u_{\varphi(n)}(t_0)\|_{L^2}^2.$$

Then,

$$\|J_N(\theta_q.\ (u(t_0 \pm \varepsilon) - u(t_0)))\|_{L^2}^2 \leq 2(\|u^0\|_{L^2}^2 - \mathrm{Re}\langle u(\varepsilon);\ u^0\rangle_{L^2})\exp(2\lambda_0 t_0).$$

By applying the monotone convergence theorem in the order $N \to \infty$ and $q \to \infty$, we obtain

$$\|u(t_0 \pm \varepsilon, \cdot) - u(t_0, \cdot)\|_{L^2}^2 \leq 2(\|u^0\|_{L^2}^2 - \mathrm{Re}\langle u(\varepsilon);\ u^0\rangle_{L^2})\exp(2\lambda_0 t_0).$$

Using the continuity at 0 and make $\varepsilon \to 0$, we obtain the continuity at t_0.

3.3 Uniqueness of the solution

Let u and v be two solutions of (S) in the space

$$C_b(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3)) \cap \mathcal{E}_b.$$

The function $w = u - v$ satisfies the following:

$$\partial_t w - \Delta w + a(\big(e^{b|u|^2} - 1\big)u - \big(e^{b|v|^2} - 1\big)v) = -\nabla(p - \tilde{p}) + w\cdot\nabla w - w\cdot\nabla u - u\cdot\nabla w.$$

Taking the scalar product in L^2 with w, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + a\langle(\big(e^{b|u|^2} - 1\big)u - \big(e^{b|v|^2} - 1\big)v);\ w\rangle_{L^2} = -\langle w\cdot\nabla u;\ w\rangle_{L^2}.$$

The idea is to lower the term $\langle((e^{b|u|^2} - 1)u - (e^{b|v|^2} - 1)v);\ w\rangle_{L^2}$ with the help of Lemma 2.3, and then divide the term find into two equal pieces, one to absorb the nonlinear term and the other is used in the last inequality.

By using inequality (2.2), we obtain

$$\langle(\big(e^{b|u|^2} - 1\big)u - \big(e^{b|v|^2} - 1\big)v);\ w\rangle_{L^2} \geq \frac{1}{2}\int_{\mathbb{R}^3}(\big(e^{b|u|^4} - 1\big) + \big(e^{b|v|^4} - 1\big))|w|^2 \geq \frac{1}{2}\int_{\mathbb{R}^3}\big(e^{b|u|^4} - 1\big)|w|^2.$$

Moreover, we have

$$|\langle w\cdot\nabla u;\ w\rangle_{L^2}| = |\langle \mathrm{div}\ (w\otimes u);\ w\rangle_{L^2}| = |\langle w\otimes u;\ \nabla w\rangle_{L^2}| \leq \int_{\mathbb{R}^3}|w|\cdot|u|\cdot|\nabla w| \leq \frac{1}{2}\int_{\mathbb{R}^3}|w|^2|u|^2 + \frac{1}{2}\|\nabla w\|_{L^2}^2$$

Combining the above inequalities, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + a\int_{\mathbb{R}^3}\big(e^{b|u|^2} - 1\big)|w|^2 \leq \int_{\mathbb{R}^3}|w|^2|u|^2.$$

By using Lemma 2.5 and the set

$$A_t = \Big\{x \in \mathbb{R}^3 / a\big(e^{b|u|^2} - 1\big) < |u(t,x)|^2\Big\},$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 \leq \lambda_0\|w\|_{L^2}^2.$$

and Gronwall lemma provides

$$\|w\|_{L^2}^2 \leq \|w^0\|_{L^2}^2 e^{\lambda_0 t}.$$

As $w^0 = 0$, then $w = 0$ and $u = v$, which implies uniqueness.

3.4 Asymptotic study of the global solution

In this subsection, we prove the asymptotic behavior (1.2). Here, we use a modified version of the Benameur-Selmi method [12]. The idea is to write the nonlinear term of exponential type as follows:

$$a\Big(e^{b|u|^2}-1\Big)u = a\Big(e^{b|u|^2}-1-b|u|^2\Big)u + a|u|^2u.$$

The first term can be treated by the same method as in [13]. So it remains to deal with the second term.

For this, let $\varepsilon > 0$ be positive real number. For $\delta > 0$, put the following functions:

$$v_\delta = \mathcal{F}^{-1}(\mathbf{1}_{B(0,\delta)}(\xi)\widehat{u}(\xi)), \;\; w_\delta = u - v_\delta.$$

We have

$$v_\delta = \sum_{k=1}^{4} f_{\delta,k}(t),$$

where

$$f_{\delta,1} = e^{t\Delta}v_\delta^0, \quad v_\delta^0 = \mathcal{F}^{-1}(\mathbf{1}_{B(0,\delta)}(\xi)\widehat{u^0}(\xi)),$$

$$f_{\delta,2} = -\int_0^t e^{(t-z)\Delta}\mathbf{1}_{B(0,\delta)}(D)\mathbb{P}\,\mathrm{div}\,(u\otimes u),$$

$$f_{\delta,3} = -a\int_0^t e^{(t-z)\Delta}\mathbf{1}_{B(0,\delta)}(D)\mathbb{P}\Big(e^{b|u|^2}-1-b|u|^2\Big)u,$$

$$f_{\delta,4} = -\int_0^t e^{(t-z)\Delta}\mathbf{1}_{B(0,\delta)}(D)\mathbb{P}(|u|^2u).$$

- We have

$$\|f_{\delta,1}(t)\|_{L^2} \le \|v_\delta^0\|_{L^2}.$$

As $\lim_{\delta\to 0}\|v_\delta^0\|_{L^2} = 0$, then there is $\delta_1 > 0$ such that

$$\sup_{t\ge 0}\|f_{\delta,1}(t)\|_{L^2} < \varepsilon/8, \quad \forall 0<\delta<\delta_1. \tag{3.5}$$

- We have

$$\begin{aligned}
\|f_{\delta,2}(t)\|_{H^{-1/4}} &\le \int_0^t \|e^{(t-z)\Delta}\mathbf{1}_{B(0,\delta)}(D)\mathbb{P}\,\mathrm{div}\,(u\otimes u)\|_{H^{-1/4}}\\
&\le \int_0^t \|\mathbf{1}_{B(0,\delta)}\mathrm{div}\,(u\otimes u)\|_{H^{-1/4}}\\
&\le \int_0^t \|\mathbf{1}_{B(0,\delta)}|D|\,(u\otimes u)\|_{H^{-1/4}}\\
&\le \int_0^t \|\mathbf{1}_{B(0,\delta)}|D|\,(u\otimes u)\|_{H^{-1/4}}\\
&\le \delta^{1/4}\int_0^t \|\mathbf{1}_{B(0,\delta)}|D|^{3/4}\,(u\otimes u)\|_{\dot H^{-1/4}}\\
&\le \delta^{1/4}\int_0^t \||D|^{3/4}\,(u\otimes u)\|_{\dot H^{-1/4}}
\end{aligned}$$

$$\begin{aligned}
&\leq \delta^{1/4}\int_0^t \|(u\otimes\ u)\|_{\dot H^{1/2}}\\
&\leq C\delta^{1/4}\int_0^t \|u\|^2_{\dot H^1},\ \ (s_1+s_2=2, s_1=s_2=1)\\
&\leq C\|u^0\|^2_{L^2}\delta^{1/4}.
\end{aligned}$$

However, if

$$\|f_{\delta,2}(t)\|_{L^2}\leq c_0(1+\delta^2)^{1/4}\|f_{\delta,2}(t)\|_{H^{-1/4}},$$

then

$$\|f_{\delta,2}(t)\|_{L^2}\leq c_0C\|u^0\|^2_{L^2}(1+\delta^2)^{1/4}\delta^{1/4}.$$

Then there is $\delta_2>0$ such that

$$\sup_{t\geq 0}\|f_{\delta,2}(t)\|_{L^2}<\varepsilon/8,\ \ \forall 0<\delta<\delta_2. \tag{3.6}$$

- We have

$$\begin{aligned}
\|f_{\delta,3}(t)\|_{H^{-2}}&\leq\int_0^t \|e^{(t-z)\Delta}\mathbf{1}_{B(0,\delta)}(D)\mathbb{P}\Big(e^{b|u|^2}-1-b|u\ |^2\Big)u\|_{H^{-2}}\\
&\leq\int_0^t \|\mathbf{1}_{B(0,\delta)}(D)\Big(e^{b|u|^2}-1-b|u\ |^2\Big)u\|_{H^{-2}}\\
&\leq R_\delta\int_0^t \|\Big(e^{b|u|^2}-1-b|u\ |^2\Big)u\|_{L^1},
\end{aligned}$$

where

$$R_\delta=\|\mathbf{1}_{B(0,\delta)}(D)\|_{H^{-2}}=\left(\int_{B(0,\delta)}\frac{1}{(1+|\xi\ |^2)^2}\mathrm{d}\xi\right)^{1/2}\leq\left(\int_{B(0,\delta)}\mathrm{d}\xi\right)^{1/2}=c_0\delta^{3/2}.$$

By using the elementary inequality with $M_b>0$

$$\Big(e^{bz^2}-1-bz^2\Big)z\leq M_b\Big(e^{bz^2}-1\Big)z^2,\quad \forall z\geq 0,$$

we obtain

$$\int_0^t \|\Big(e^{b|u|^2}-1-b|u\ |^2\Big)u\|_{L^1}\leq M_b\int_0^t \|\Big(e^{b|u|^2}-1\Big)|u\ |^2\|_{L^1}\leq (2a)^{-1}M_b\|u^0\|^2_{L^2}.$$

Combining the above inequalities, we obtain

$$\|f_{\delta,3}(t)\|_{H^{-2}}\leq c_0(2a)^{-1}M_b\|u^0\|^2_{L^2}\delta^{3/2}.$$

However, if

$$\|f_{\delta,3}(t)\|_{L^2}\leq c_0(1+\delta^2)^2\|f_{\delta,2}(t)\|_{H^{-2}},$$

then

$$\|f_{\delta,3}(t)\|_{L^2}\leq c_0(2a)^{-1}M_b\|u^0\|^2_{L^2}(1+\delta^2)^2\delta^{3/2}.$$

Then there is $\delta_3 > 0$ such that

$$a \sup_{t\geq 0} \|f_{\delta,3}(t)\|_{L^2} < \varepsilon/8, \quad \forall 0 < \delta < \delta_3. \tag{3.7}$$

- By using Lemma 2.2 with the well choice of s_1 and s_2, we obtain

$$\begin{aligned}
\|f_{\delta,4}(t)\|_{H^{-1/2}} &\leq \int_0^t \|e^{(t-z)\Delta}\mathbf{1}_{B(0,\delta)}(D)\mathbb{P}|u|^2u\|_{H^{-1/2}} \\
&\leq \int_0^t \|\mathbf{1}_{B(0,\delta)}(D)|u|^2u\|_{H^{-1/2}} \\
&\leq \int_0^t \|\mathbf{1}_{B(0,\delta)}(D)|u|^2u\|_{\dot{H}^{-1/2}} \\
&\leq \int_0^t \|\mathbf{1}_{B(0,\delta)}(D)|D|^{1/2}|D|^{-1/2}|u|^2u\|_{\dot{H}^{-1/2}} \\
&\leq \delta^{1/2}\int_0^t \|\mathbf{1}_{B(0,\delta)}(D)|D|^{-1/2}|u|^2u\|_{\dot{H}^{-1/2}} \\
&\leq \delta^{1/2}\int_0^t \||D|^{-1/2}|u|^2u\|_{\dot{H}^{-1/2}} \\
&\leq \delta^{1/2}\int_0^t \||u|^2u\|_{\dot{H}^{-1}} \\
&\leq \delta^{1/2}C\int_0^t \|u\|_{\dot{H}^{1/2}}\||u|^2\|_{L^2}, \quad (s_1+s_2=1/2,\, s_1=0,\ s_2=1/2) \\
&\leq \delta^{1/2}C'\int_0^t \|u\|^2_{\dot{H}^{1/2}}\|u\|_{\dot{H}^1}, \quad (s_1+s_2=3/2,\ s_1=1/2,\ s_2=1), \\
&\leq \delta^{1/2}C'\int_0^t \|u\|_{L^2}\|u\|^2_{\dot{H}^1}, \quad (\text{by interpolation}), \\
&\leq \delta^{1/2}C''\|u^0\|_{L^2}\int_0^t \|\nabla u\|^2_{L^2} \\
&\leq \delta^{1/2}C''\|u^0\|^3_{L^2}.
\end{aligned}$$

However, if

$$\|f_{\delta,4}(t)\|_{L^2} \leq c_0(1+\delta^2)^{1/2}\|f_{\delta,2}(t)\|_{H^{-1/2}},$$

then

$$\|f_{\delta,4}(t)\|_{L^2} \leq c_0C''\|u^0\|^3_{L^2}(1+\delta^2)^{1/2}\delta^{1/2}.$$

Then there is $\delta_4 > 0$ such that

$$\sup_{t\geq 0} \|f_{\delta,4}(t)\|_{L^2} < \varepsilon/8, \quad \forall 0 < \delta < \delta_4. \tag{3.8}$$

Combining the equations (3.5)–(3.8), we obtain

$$\sup_{t\geq 0}\|v_{\delta_0}(t)\|_{L^2} < \varepsilon/2,\ \ \delta_0 = \frac{1}{2}\min_{1\leq i\leq 4}\delta_i. \tag{3.9}$$

On the other hand, we have

$$\int_0^\infty \|w_{\delta_0}(t)\|_{L^2}^2 dt \leq \delta_0^{-2}\int_0^\infty \|\nabla w_{\delta_0}(t)\|_{L^2}^2 \mathrm{d}t \leq \delta_0^{-2}\int_0^\infty \|\nabla w_{\delta_0}(t)\|_{L^2}^2 \mathrm{d}t \leq \delta_0^{-2}\|u^0\|_{L^2}^2.$$

As $(t \to \|w_{\delta_0}(t)\|_{L^2}^2)$ is continuous, then there is a time $t_0 \geq 0$ such that

$$\|w_{\delta_0}(t_0)\|_{L^2}^2 < \varepsilon/2. \tag{3.10}$$

Combining inequalities (3.9) and (3.10), we obtain

$$\|u(t_0)\|_{L^2} \leq \|v_{\delta_0}(t_0)\|_{L^2} + \|w_{\delta_0}(t_0)\|_{L^2} < \varepsilon.$$

As $(t \to \|w_{\delta_0}(t)\|_{L^2}^2)$ is decreasing, then

$$\|u(t)\|_{L^2} < \varepsilon, \quad \forall t \geq t_0.$$

which completes the proof.

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