Math 106
Integral Calculus
Parametric Equations and Polar Coordinates

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Parametric Equations

Parametric Equations

Definition

A plane curve is a set of ordered pairs $(f(t), g(t))$, where f and g are continuous on an interval I.

Parametric Equations

Definition

Let C be a curve consists of all ordered pairs $(f(t), g(t))$, where f and g are continuous on an interval I. The equations

$$x = f(t), \qquad y = g(t)$$

for $t \in I$ are parametric equations for C with parameter t.

Example

Sketch the graph of C where C is the curve

$$x = t, \qquad y = t^2, \qquad -1 \leq t \leq 2$$

Example

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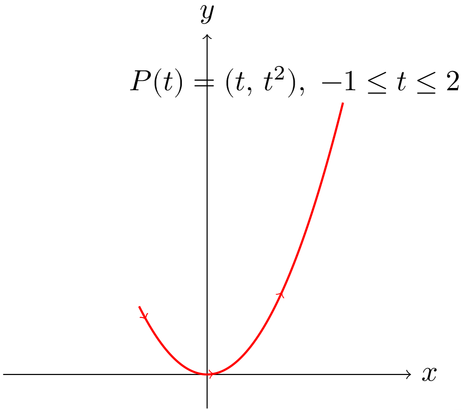


Example

Sketch the graph of C where C is the curve

$$x = \sqrt{t}, \qquad y = t, \qquad t \geq 0$$

Parametric Equations

Example

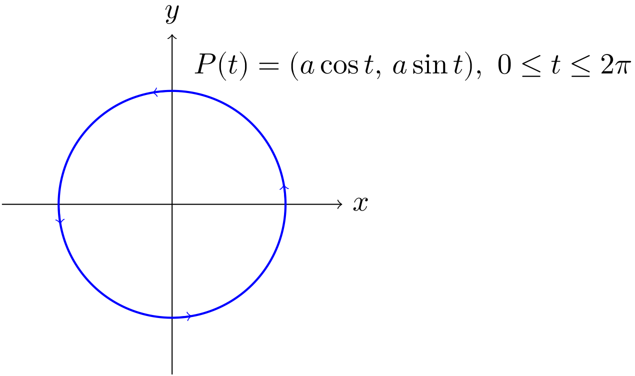
Sketch the graph of C where C is the curve

$$x = a\cos t, \qquad y = a\sin t, \qquad a > 0, \qquad t \geq 0$$

Example

Sketch the graph of C where C is the curve

$$x = a\cos t, \qquad y = a\sin t, \qquad a > 0, \qquad t \geq 0$$



Example

Sketch the graph of C where C is the curve

$$x = -2 + t^2, \qquad y = 1 + 2t^2 \qquad t \in \mathbb{R}$$

Arc Length and Surface Area

Slope of a Curve

Theorem

If a smooth curve C is given parametrically by $x = f(t), y = g(t)$ then the slope $\frac{dy}{dx}$ of the tangent line to C at a point $P(x, y)$ is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \frac{dx}{dt} \neq 0$$

Slope of a Curve

Example

Find the slope of the tangent line of C where C is the curve

$$x = 2t, \qquad y = t^2 - 1 \qquad -1 \leq t \leq 2$$

Slope of a Curve

Example

Find the equation of the tangent line at $t = 2$ of C where C is the curve

$$x = t^3 - 3t, \qquad y = t^2 - 5t - 1, \qquad t \geq 0$$

Second Derivative

Definition

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(y') = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$$

Note that

$$\frac{d^2y}{dx^2} \neq \frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}}$$

Slope of a Curve

Example

Find $\frac{d^2y}{dx^2}$ of the curve C

$$x = 2t - 3, \qquad y = t^2 + 4t, \qquad t \in \mathbb{R}$$

Arc Length

Theorem

If a smooth curve C is given parametrically by $x = f(t), y = g(t)$; $a \leq t \leq b$ and C does not intersect itself, except possibly for $t = a$ and $t = b$. Then the length L of C is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Arc Length

Example

Find the arc length of the curve

$$x = e^t \cos t, y = e^t \sin t, 0 \leq t \leq \pi$$

Arc Length

Exam problem

Find the arc length of the parametric curve $x = \sin t - t\cos t$, $y = \cos t + t\sin t$, $0 \leq t \leq 2$

Surface Area

Theorem

If a smooth curve C is given parametrically by $x = f(t), y = g(t)$; $a \leq t \leq b$ and C does not intersect itself, except possibly for $t = a$ and $t = b$. If $g(t) \geq 0$, then the area S of the surface of revolution obtained by revolving C about the $x-$axis is

$$S = \int_a^b 2\pi g(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Theorem

If $f(t) \geq 0$, then the area S of the surface of revolution obtained by revolving C about the $y-$axis is

$$S = \int_a^b 2\pi f(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Surface Area

Example

Find the surface area of the solid obtained by revolving the curve

$$x = \sin t,\ y = \cos t,\ 0 \le t \le \frac{\pi}{2}$$

about the x-axis

Exam problem

Find the surface area obtained by revolving the curve $x = t^3$, $y = 3t + 1, 0 \leq t \leq 1$ about the y-axis

Polar Coordinates

Polar Coordinates

We normally use **Cartesian (Rectangular) Coordinates** to determine points $P(x, y)$.

The **Polar Coordinate** system is an alternative to the cartesian coordinate system which consists of a pole and a polar axis. Each point P on a plane is determined by a distance r from the pole O and an angle θ from the polar axis.

$$P(r, \theta)$$

Polar Coordinates

Example

Plot the points

1. $(2, \frac{\pi}{4})$
2. $(1, 3\pi)$
3. $(2, -\frac{\pi}{6})$
4. $(-3, \frac{2\pi}{3})$

Polar and Rectangular Coordinates

The rectangular coordinates (x, y) and polar coordinates (r, θ) of a point P are related as follows:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$r^2 = x^2 + y^2$$
$$\tan\theta = \frac{y}{x}, \qquad x \neq 0$$

Polar and Rectangular Coordinates

Example

Convert from polar coordinates to rectangular coordinates

1. $(2, \frac{\pi}{4})$
2. $(3, \frac{3\pi}{4})$
3. $(1, \pi)$

Polar and Rectangular Coordinates

Example

Convert from rectangular coordinates to polar coordinates

1. $(1, 1)$
2. $(-2, 2)$
3. $(\sqrt{3}, -1)$

Functions in Polar Coordinates

Example

Plot the functions

1. $r = 1$
2. $\theta = \frac{\pi}{2}$
3. $r = \theta$

Circle

Example

Plot the function

$$r = 2\cos\theta$$

Cardioid

Example

Plot the function

$$r = 1 + \sin\theta$$

This Called **cardioid**

Important Functions

$$r = a$$

$$r = a\sin\theta, \qquad r = -a\sin\theta$$

$$r = a\cos\theta, \qquad r = a\cos\theta$$

$$r = a(1+\cos\theta), \qquad r = a(1-\cos\theta)$$

$$r = a(1+\sin\theta), \qquad r = a(1-\sin\theta)$$

Integrals In Polar Coordinates

Area

Area

Theorem

If f is continuous and $f(\theta) \geq 0$ on $[a, b]$, where $0 \leq a < b \leq 2\pi$, then the area A of the region bounded by the graph of $r = f(\theta)$, $\theta = a$, and $\theta = b$ is

$$A = \int_a^b \frac{1}{2} f(\theta)^2 \, d\theta = \int_a^b \frac{1}{2} r^2 \, d\theta$$

Area

Example

Find the area of the region bounded by

$$r = 2 + 2\cos\theta$$

Area

Example

Find the area of the region inside $r = 2 + 2\cos\theta$, and outside $r = 3$.

Exam problem

Sketch the region inside $r = 2 + 2\sin\theta$ and outside $r = 2 - 2\sin\theta$ and find its area

Exam problem

Sketch the region inside $r = 3\sin\theta$ and outside $r = 3 - 3\sin\theta$ and find its area

Arc Length

To find the arc length of $r = f(\theta)$ from $\theta = a$ to $\theta = b$, we use parametric equations

$$x = f(\theta)\cos\theta, \qquad y = f(\theta)\sin\theta, \qquad a \le \theta \le b$$

Arc Length

Theorem

The arc length of $f(\theta)$ *from* $\theta = a$, *to* $\theta = b$ *is*

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Arc Length

Example

Find the length of the cardioid $r = 1 + \cos\theta$

Surface of Revolution

Theorem

The surface of revolution of $f(\theta)$ from $\theta = a$, to $\theta = b$ about the polar axis is

$$S = \int_a^b 2\pi r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

and about the line $\theta = \frac{\pi}{2}$ is

$$S = \int_a^b 2\pi r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example

Find the area of the surface of revolution of $r = e^{\frac{\theta}{2}}$ from $\theta = 0$, to $\theta = \pi$ about the polar axis.