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10 Homomorphisms

Exercise 1: Prove that the mapping given in Example 2 is a homomorphism.

Example 2: Let R^* be the group of nonzero real numbers under multiplication. Then the determinant mapping $A \to \det A$ is a homomorphism from $GL(2, R)$ to R^*. The kernel of the determinant mapping is $SL(2, R)$.

Solution: Note that $\det(AB) = (\det A)(\det B)$.

Exercise 2: Prove that the mapping given in Example 3 is a homomorphism.

Example 3: The mapping ϕ from R^* to R^*, defined by $\phi(x) = |x|$, is a homomorphism with $\text{Ker } \phi = \{1, -1\}$.

Solution: Note that $|xy| = |x||y|$ for all $x, y \in R^*$.

Exercise 3: Prove that the mapping given in Example 4 is a homomorphism.

Example 4: Let $R[x]$ denote the group of all polynomials with real coefficients under addition. For any f in $R[x]$, let f' denote the derivative of f. Then the mapping $f \to f'$ is a homomorphism from $R[x]$ to itself. The kernel of the derivative mapping is the set of all constant polynomials.

Solution: Note that $(f + g)' = f' + g'$.

Exercise 4: Prove that the mapping given in Example 12 is a homomorphism.

Example 12: The mapping from S_n to Z_2 that takes an even permutation to 0 and an odd permutation to 1 is a homomorphism.

Solution: $\phi(EE) = \phi(E) = 0 = 0 + 0 = \phi(E) + \phi(E)$. $\phi(EO) = \phi(O) = 1 = 0 + 1 = \phi(E) + \phi(O)$. The other cases are similar.

Exercise 7: If ϕ is a homomorphism from G to H and σ is a homomorphism from H to K, show that $\sigma\phi$ is a homomorphism from G to K. How are $\text{Ker } \phi$ and $\text{Ker } \sigma\phi$ related? If ϕ and σ are onto and G is finite, describe $[\text{Ker } \sigma\phi : \text{Ker } \phi]$**** in terms of **|H|**$and$**|K|.**

Solution: $(\sigma\phi)(g_1 g_2) = \sigma(\phi(g_1 g_2)) = \sigma(\phi(g_1)\phi(g_2)) = \sigma(\phi(g_1))\sigma(\phi(g_2)) = (\sigma\phi)(g_1)(\sigma\phi)(g_2)$. It follows from Theorem 10.3 that $|G/\text{Ker }\phi| = |H|$ and $|G/\text{Ker }\sigma\phi| = |K|$. Thus, $[\text{Ker }\sigma\phi : \text{Ker }\phi] = |\text{Ker }\sigma\phi/\text{Ker }\phi| = |H|/|K|$.

Exercise 11: Prove that $(Z \oplus Z)/(\langle(a,0)\rangle \times \langle(0,b)\rangle)$ is isomorphic to $Z_a \oplus Z_b$.

Solution: The mapping $\phi: Z \oplus Z \to Z_a \oplus Z_b$ given by $\phi((x,y)) = (x \mod a, y \mod b)$ is operation preserving. If $(x,y) \in \text{Ker }\phi$, then $x \in \langle a\rangle$ and $y \in \langle b\rangle$. So, $(x,y) \in \langle(a,0)\rangle \times \langle(0,b)\rangle$. Conversely, every element in $\langle(a,0)\rangle \times \langle(0,b)\rangle$ is in $\text{Ker }\phi$. So, by Theorem 10.3, $(Z \oplus Z)/(\langle(a,0)\rangle \times \langle(0,b)\rangle) \cong Z_a \oplus Z_b$.

Exercise 12: Suppose that k is a divisor of n. Prove that $Z_n/\langle k\rangle \cong Z_k$.

Solution: Since the factor group of a cyclic group is cyclic and $|Z_n/\langle a\rangle| = n/|a|$, we have $Z_n/\langle k\rangle$ is isomorphic to $Z_{n/k}$. Wait, this should be Z_k. Let me recalculate: If $k | n$, then $\langle k\rangle$ has order n/k. So $|Z_n/\langle k\rangle| = n/(n/k) = k$, thus $Z_n/\langle k\rangle \cong Z_k$.

Exercise 13: Prove that $(A \times B)/(A \times \{e\}) \cong B$.

Solution: The mapping $(a,b) \to b$ is a homomorphism from $A \times B$ onto B with kernel $A \times \{e\}$. So, by Theorem 10.3, $(A \times B)/(A \times \{e\}) \cong B$.

Exercise 14: Explain why the correspondence $x \to 3x$ from Z_{12} to Z_{10} is not a homomorphism.

Solution: Observe that since 1 has order 12, $|\phi(1)| = |3|$ must divide 12. But in Z_{10}, $|3| = 10$.

Exercise 15: Suppose that ϕ is a homomorphism from Z_{30} to Z_{30} and $\text{Ker }\phi = \{0, 10, 20\}$. If $\phi(23) = 9$, determine all elements that map to 9.

Solution: By property 6 of Theorem 10.1, we know $\phi^{-1}(9) = 23 + \text{Ker }\phi = \{23, 3, 13\}$.

Exercise 16: Prove that there is no homomorphism from $Z_8 \oplus Z_2$ onto $Z_4 \oplus Z_4$.

Solution: Suppose ϕ is such a homomorphism. By Theorem 10.3, $|\text{Ker }\phi| = 2$. Let $\phi(1,0) = (a,b)$. Then $\phi(4,0) = 4\phi(1,0) = 4(a,b) = (4a,4b) = (0,0)$. But then $\text{Ker }\phi$ contains an element of order 4.

Exercise 19: Suppose that there is a homomorphism ϕ from Z_{17} to some group and that ϕ is not one-to-one. Determine ϕ.

Solution: Since $|\mathrm{Ker}\ \phi|$ is not 1 and divides 17, ϕ is the trivial map.

Exercise 21: If ϕ is a homomorphism from Z_{30} onto a group of order 5, determine the kernel of ϕ.

Solution: By Theorem 10.3 we know that $|Z_{30}/\mathrm{Ker}\ \phi| = 5$. So, $|\mathrm{Ker}\ \phi| = 6$. The only subgroup of Z_{30} of order 6 is $\langle 5 \rangle$.

Exercise 23: Let ϕ be a homomorphism from a finite group G to $\bar{G}$. If H is a subgroup of $\bar{G}$, give a formula for $|\phi^{-1}(H)|$ in terms of $|H|$ and ϕ.

Solution: $|\phi^{-1}(H)| = |H| \cdot |\mathrm{Ker}\ \phi|$.

Exercise 38: Let α be a homomorphism from G_1 to H_1 and β be a homomorphism from G_2 to H_2. Determine the kernel of the homomorphism γ from $G_1 \oplus G_2$ to $H_1 \oplus H_2$ defined by $\gamma(g_1, g_2) = (\alpha(g_1), \beta(g_2))$.

Solution: $\mathrm{Ker}\ \gamma = \mathrm{Ker}\ \alpha \times \mathrm{Ker}\ \beta$.

Exercise 49 (Second Isomorphism Theorem): If K is a subgroup of G and N is a normal subgroup of G, prove that $K/(K \cap N)$ is isomorphic to KN/N.

Solution: Consider the mapping ϕ from K to KN/N given by $\phi(k) = kN$. Since $\phi(kk') = kk'N = kNk'N = \phi(k)\phi(k')$ and $kN \in KN/N$, ϕ is a homomorphism. Moreover, $\mathrm{Ker}\ \phi = K \cap N$. So, by Theorem 10.3, $K/(K \cap N) \cong KN/N$.

Exercise 50 (Third Isomorphism Theorem): If M and N are normal subgroups of G and $N \subseteq M$, prove that $(G/N)/(M/N) \cong G/M$. Think of this as a form of "cancelling out" the N in the numerator and denominator.

Solution: Let ϕ be the natural mapping from G to G/N given by $\phi(g) = gN$. Consider the natural mapping ψ from G/N to $(G/N)/(M/N)$ given by $\psi(gN) = (gN)(M/N) = gM$. Then $\psi\phi$ is a homomorphism from G onto $(G/N)/(M/N)$ with kernel M. By Theorem 10.3, $G/M \cong (G/N)/(M/N)$.

Exercise 71: Prove that for any two primes p and q with $p < q$ where p does not divide $q - 1$, a group of order pq is cyclic.

Solution: Mimic Example 18. (Example 18 shows that for a group G of order 35, by considering the normalizer and centralizer of a Sylow 7-subgroup H, we can use the N/C Theorem to show that $C(H) = G$, which means H is in the center, so G is Abelian.

Since G has a unique subgroup of order 7 and a unique subgroup of order 5, G must be cyclic. The same argument works for order pq when $p \nmid q - 1$.)