MOMENT GENERATING FUNCTIONS

Q1) Suppose independent r.v.'s X and Y are such that $M_{X+Y}(t) = \frac{e^{2t}-1}{2t-t^2}$
If $f(x) = \lambda e^{-\lambda x}$; $x > 0$, what is the *mgf* of Y.

Solution :

X and Y are independent $M_{X+Y}(t) = M_X(t)M_Y(t) \Rightarrow M_Y(t) = \frac{M_{X+Y}(t)}{M_X(t)}$

So, we need to find *mgf* of X which is

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx}\left(\lambda e^{-\lambda x}\right) dx = \lambda \int_0^\infty e^{-(\lambda-t)x} dx = \frac{-\lambda}{(\lambda-t)}\left[e^{-(\lambda-t)x}\right]_0^\infty$$

$$M_X(t) = \frac{\lambda}{(\lambda-t)} \quad , \qquad \lambda - t > 0 \;\Rightarrow\; \lambda > t$$

$$M_Y(t) = M_{X+Y}(t)\frac{1}{M_X(t)} = \frac{e^{2t}-1}{2t-t^2}\,\frac{1}{\frac{\lambda}{(\lambda-t)}} = \frac{e^{2t}-1}{2t-t^2}\,\frac{(\lambda-t)}{\lambda}$$

Q2) Find the moment generating function of X If you know that $f(x) = 2e^{-2x}, x > 0$
Solution : H.W

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx}\left(2e^{-2x}\right) dx = 2\int_0^\infty e^{-(2-t)x}\; dx = \frac{-2}{(2-t)}\left[e^{-(2-t)x}\right]_0^\infty$$

$$= \frac{-2}{(2-t)}(0-1) = \frac{2}{2-t} \qquad , 2-t > 0 \Rightarrow 2 > t$$

Q3) A r.v. has $f(x) = \frac{1}{2}\, e^{-|x|}$; $for -\infty < x < \infty$.

a. Show that its *mgf* is given by $M_X(t) = \frac{1}{1-t^2}$ $for \; -1 < t < 1$.

b. Using the *mgf*, find E(X) and V(X).

Solution :

$$\boldsymbol{f(x)} = \begin{cases} \frac{1}{2}\, e^{-x} & , x > 0 \\ \frac{1}{2}\, e^{x} & , x < 0 \end{cases}$$

a) $M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx}\left(\frac{1}{2}e^{-x}\right) dx + \int_{-\infty}^0 e^{tx}\left(\frac{1}{2}e^{x}\right) dx$

$$= \frac{1}{2}\int_0^\infty e^{-(1-t)x}\; dx + \frac{1}{2}\int_{-\infty}^0 e^{(1+t)x}\; dx \qquad (*)$$

$$\frac{1}{2}\int_0^\infty e^{-(1-t)x}\; dx = \frac{-1}{2(1-t)}\left[e^{-(1-t)x}\right]_0^\infty = \frac{1}{2(1-t)} \qquad (1)$$

$$\frac{1}{2}\int_{-\infty}^0 e^{(1+t)x}\; dx = \frac{1}{2(1+t)}\left[e^{(1+t)x}\right]_{-\infty}^0 = \frac{1}{2(1+t)} \qquad (2)$$

Substitute (1) and (2) in (*) we get

$$M_X(t) = \frac{1}{2(1-t)} + \frac{1}{2(1+t)} = \frac{1}{2}\left[\frac{1+t+1-t}{(1-t)(1+t)}\right] = \frac{1}{2}\left(\frac{2}{1+t-t-t^2}\right) = \frac{1}{1-t^2}$$

b) $M_X(t) = (1-t^2)^{-1}$

$M_X'(t) = (-1)(1-t^2)^{-2}(-2t) = 2t(1-t^2)^{-2} \Rightarrow M_X'(0) = \boldsymbol{E(X) = 0}$

$M_X''(t) = 2t(-2)(1-t^2)^{-3}(-2t) + 2(1-t^2)^{-2} \Rightarrow M_X''(0) = \boldsymbol{E(X^2) = 2}$

$$\therefore \boldsymbol{V(X) = E(X^2) - [E(X)]^2 = 2 - 0 = 2}$$

Q4) If X has $f(x) = \frac{3}{2}\ x^2\ ,\ -1 < x < 1$

a. Find *mgf* of X.

Solution :

a) $M_X(t) = E(e^{tx}) = \int_{-1}^{1} e^{tx}\left(\frac{3}{2}x^2\right) dx = \frac{3}{2}\int_{-1}^{1} x^2 e^{tx}\ dx \qquad (1)$

by use Integration by Parts $\int_a^b u\, dv = [uv]_a^b - \int_a^b v\, du$

$$u = x^2 \rightarrow du = 2x\, dx \quad dv = e^{tx}\, dx \rightarrow \quad v = \frac{1}{t}e^{tx}$$

$$M_X(t) = \frac{3}{2}\left[\frac{1}{t}[x^2 e^{tx}]_{-1}^{1} - \int_{-1}^{1}\frac{1}{t}2x\, e^{tx} dx\right] = \frac{3}{2}\left[\frac{1}{t}(e^t - e^{-t}) - \frac{2}{t}\int_{-1}^{1}\boldsymbol{x\, e^{tx} dx}\right] \qquad (2)$$

by use Integration by Parts: $u = x \rightarrow du = dx\ ; \quad dv = e^{tx}\, dx \rightarrow \quad v = \frac{1}{t}e^{tx}\, dx$

$$\int_{-1}^{1}\boldsymbol{x\, e^{tx} dx} = \left[\frac{1}{t}xe^{tx}\right]_{-1}^{1} - \int_{-1}^{1}\frac{1}{t}e^{xt}\, dx = \frac{1}{t}(e^t + e^{-t}) - \frac{1}{t^2}[e^{tx}]_{-1}^{1}$$

$$= \frac{1}{t}(e^t + e^{-t}) - \frac{1}{t^2}(e^t - e^{-t}) \qquad (3)$$

Substitute (3) in (2) we get

$$\boldsymbol{M_X(t)} = \frac{3}{2}\left[\frac{1}{t}(e^t - e^{-t}) - \frac{2}{t}\left[\frac{1}{t}(e^t + e^{-t}) - \frac{1}{t^2}(e^t - e^{-t})\right]\right]$$

$$= \frac{3}{2}\left[\frac{1}{t}(e^t - e^{-t}) - \frac{2}{t^2}(e^t + e^{-t}) + \frac{2}{t^3}(e^t - e^{-t})\right]$$

$$= \frac{3}{2}\left(\frac{1}{t} + \frac{2}{t^3}\right)(e^t - e^{-t}) - \frac{3}{t^2}(e^t + e^{-t})$$

Q5) Suppose X is a r.v. for which the *mgf* is $M_X(t) = \frac{1}{4}(3e^t + e^{-t})\,, -\infty < t < \infty$.

a. Find the mean and variance of X.

Solution : H.W

a) $E(X) = M'_X(t)|_{t=0} = \frac{1}{4}(3e^t - e^{-t})\Big|_{t=0} = \frac{1}{4}(3-1) = \frac{2}{4} = \frac{1}{2}$

$$E(X^2) = M''_X(t)|_{t=0} = \frac{1}{4}(3e^t + e^{-t})\Big|_{t=0} = \frac{1}{4}(3+1) = 1$$

$$\therefore\ V(Z) = E(Z^2) - [E(Z)]^2 = 1 - \frac{1}{4} = \frac{3}{4}$$

Q6) X and Y are independent and identically distributed with $M(t) = e^{3t+t^2}$.

Find the *mgf* of $Z = 2X - 3Y + 4$ and use it to find the mean and variance of Z.

Solution :

$$M_{aX+b}(t) = e^{bt}E(e^{atX}) = e^{bt}M_X(at)$$
$$M_{X+Y}(t) = E(e^{tX+tY}) = M_X(t)M_Y(t)$$

As X and Y are independent and identically distributed, then

$M_Z(t) = M_{2X-3Y+4}(t) = M_{2X}(t)M_{-3Y+4}(t)$

$= M_X(2t)[e^{4t}\,M_Y(-3t)] = e^{6t+4t^2}e^{4t}\,e^{-9t+9t^2} \quad \Rightarrow \quad \boldsymbol{M_Z(t) = e^{t+13\,t^2}}$

$M'_Z(t) = (1+26t)e^{t+13\,t^2} \rightarrow M'_Z(0) = \boldsymbol{E(Z) = 1}$

$M''_Z(t) = 26\,e^{t+13\,t^2} + (1+26t)^2e^{t+13\,t^2} \rightarrow M''_Z(0) = E(Z^2) = 27$

So, $\boldsymbol{V(Z)} = E(Z^2) - [E(Z)]^2 = \boldsymbol{27 - 1 = 26}$

Q7) Suppose X has $M_X(t) = e^{3t+t^2}$.

Find the *mgf* of $Z = \frac{1}{4}(X - 3)$ and use it to find the mean and variance of Z.

Solution : H.W

$$M_X(t) = e^{3t+t^2}$$

$$M_Z(t) = M_{\frac{1}{4}(X-3)}(t) = M_{\frac{1}{4}X-\frac{3}{4}}(t) = e^{-\frac{3}{4}t}M_X\left(\frac{1}{4}t\right) = e^{-\frac{3}{4}t}\,e^{\frac{3}{4}t+\left(\frac{1}{4}t\right)^2}$$

$$= e^{-\frac{3}{4}t}\,e^{\frac{3}{4}t+\frac{1}{16}t^2} = e^{\frac{1}{16}t^2}$$

$$E(Z) = M'_Z(t)|_{t=0} = \frac{2}{16}t\,e^{\frac{1}{16}t^2}\Big|_{t=0} = 0$$

$$E(Z^2) = M''_Z(t)|_{t=0} = \frac{2}{16}\left[\frac{2}{16}t^2\,e^{\frac{1}{16}t^2} + e^{\frac{1}{16}t^2}\right]\Big|_{t=0} = \frac{2}{16} = \frac{1}{8}$$

$\therefore\ V(Z) = E(Z^2) - [E(Z)]^2 = \frac{1}{8}$

Q8) Let $\boldsymbol{f(x) = 1 \; ; \; 0 \leq x \leq 1}$. Use the moment generating function technique to find the moment generating function of $Y = aX + b$ where *(a)* and *(b)* are constant.

Solution :

$$M_X(t) = \int_0^1 e^{tx}\, dx = \left[\frac{e^{tx}}{t}\right]_0^1 = \frac{e^t}{t} - \frac{1}{t} \qquad \Rightarrow \boldsymbol{M_X(t) = \frac{e^t - 1}{t}}$$

$$M_Y(t) = e^{tb} M_X(at) = e^{tb}\,\frac{e^{at} - 1}{at} = \frac{e^{t(a+b)} - e^{bt}}{at} \;\; ; \;\; at > 0 \;\Leftrightarrow a > 0$$

$$Y \sim Uniform(b, a+b)$$

But if $a < 0 \Rightarrow M_Y(t) = \frac{-(e^{bt} - e^{t(a+b)})}{at} = \frac{e^{bt} - e^{t(a+b)}}{-at}$, $-a > 0$; $Y \sim Uniform(a+b, b)$

Note: continuous uniform distribution (a,b)

$f(x) = \frac{1}{b-a} \;\; a < x < b \;\; and \quad M_X(t) = \frac{e^{tb} - e^{ta}}{t(b-a)} \;\; t \neq 0$

Q9) Let $f(x) = e^{-x}$; $x > 0$, find the *mgf* of $Z = 3 - 2X$.

Solution : H.W

$$M_X(t) = \int_0^\infty e^{tx}\; e^{-x}\, dx = \int_0^\infty e^{-x(1-t)}\; dx = \left[\frac{e^{-x(1-t)}}{-(1-t)}\right]_0^\infty = 0 - \frac{1}{-(1-t)} = \frac{1}{1-t}$$

$$M_X(t) = \frac{1}{1-t}$$

$$M_Z(t) = e^{3t}\, M_X(-2t) = \frac{e^{3t}}{1+2t} \quad ,1 + 2t > 0 \Rightarrow \quad 2t > -1 \;\Rightarrow t > -\frac{1}{2}$$

Q10) *X*, *Y* and *Z* are independent r.v.'s with $X \sim Normal(1,3), Y \sim Normal(5,2)$ and the *mgf* of their sum being $M_{X+Y+Z}(t) = e^{13t + 3t^2}$. Determine the distribution of Z.

Solution :

We know that if $X \sim Normal(\mu, \sigma^2)$, then

$$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

$$X \sim N(1,3) \Rightarrow M_X(t) = e^{t + \frac{3}{2}t^2}$$

$$Y \sim N(5,2) \Rightarrow M_Y(t) = e^{5t + t^2}$$

$$M_{X+Y+Z}(t) = e^{13t + 3t^2}$$

X, *Y* and *Z* are independent r.v.'s , then $M_{X+Y+Z}(t) = M_X(t) M_Y(t)\, M_Z(t)$

$$M_Z(t) = \frac{M_{X+Y+Z}(t)}{M_X(t) M_Y(t)} = \frac{e^{13t + 3t^2}}{e^{t + \frac{3}{2}t^2}\;\; e^{5t + t^2}} = \frac{e^{13t + 3t^2}}{e^{6t + \frac{5}{2}t^2}} = e^{7t + \frac{1}{2}t^2} = e^{7t + \frac{1}{2}(1^2)\, t^2}$$

$$\therefore Z \sim N(7,1)$$

Q11) Identify the distribution of the r.v. from the moment generating function

a) $M_X(t) = \frac{1}{1-2t}$, $t < 1/2$

b) $M_X(t) = e^{3t+2t^2}$

c) X, Y independent, $M_{X+Y}(t) = \left(\frac{2}{2-t}\right)^3$, $t < \frac{1}{2}$, $Y \sim Exp(2)$

Solution:

a) $X \sim exp(\theta) \rightarrow \mu_X(t) = \frac{\theta}{(\theta-t)}$, $t < \theta$

$Therefore\ \frac{1}{(1-2t)} = \frac{\frac{1}{2}}{\frac{1}{2}-t} \rightarrow \theta = \frac{1}{2}$

b) $X \sim N(\mu, \sigma^2) \rightarrow \mu_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t}$

$Therefore\ \mu = 3\ and\ \frac{1}{2}\sigma^2 = 2 \rightarrow \sigma^2 = 4$

c) X and Y are independent

$\mu_{X+Y}(t) = \left(\frac{2}{2-t}\right)^3$; $Y = exp(\theta = 2) \Rightarrow \mu_Y(t) = \frac{2}{2-t}$

$Therefore\ \mu_{X+Y}(t) = \mu_X(t)\,\mu_Y(t)$

$\rightarrow \left(\frac{2}{2-t}\right)^3 = \mu_X(t)\left(\frac{2}{2-t}\right)$

$\rightarrow \mu_X(t) = \frac{\left(\frac{2}{2-t}\right)^3}{\left(\frac{2}{2-t}\right)} = \left(\frac{2}{2-t}\right)^2$

$Therefore\quad X \sim gamma\ (\alpha = 2, \lambda = 2)$

Note: $f(x) = \frac{\lambda}{\Gamma(\alpha)}(x\lambda)^{\alpha-1}e^{-\lambda x}$, $x > 0$ and $M_X(t) = \left(\frac{\lambda}{\lambda-t}\right)^\alpha$

Q12) X, Y independent, $M_{X+Y}(t) = \frac{e^{2t}-1}{2t-t^2}$, $X \sim Exp(2)$, what is the distribution of Y

Solution: H.W

X and Y are independent

$\mu_{X+Y}(t) = \frac{e^{2t}-1}{(2t-t^2)}$

$X \sim exp\ (\theta = 2) \rightarrow \mu_X(t) = \frac{2}{2-t}$

$$\mu_{X+Y}(t) = \mu_X(t)\ \mu_Y(t)$$

$$\rightarrow \quad \frac{(e^{2t}-1)}{(2t-t^2)} = \frac{2}{2-t}\ \mu_Y(t)$$

$$\rightarrow \mu_Y(t) = \frac{\frac{e^{2t}-1}{t(2-t)}}{\frac{2}{2-t}} = \frac{e^{2t}-1}{t(2-t)}\ \frac{2-t}{2} = \frac{(e^{2t}-1)}{2t} = \frac{e^{2t}-e^0}{t(2-0)}$$

$$Therefore \quad Y \sim continuous\ Uniform\ (a=0, b=2)$$

Note: continuous uniform distribution

$f(x) = \frac{1}{b-a}\quad a < x < b \quad and \quad M_X(t) = \frac{e^{tb}-e^{ta}}{t(b-a)} \quad t \neq 0$

Q13) If $X \sim Gamma(2,3)$ independent of $Y \sim Uniform(0,2)$, and $Z \sim Gamma(5,3)$, find mgf of $X+Y+Z$ if X, Y and Z are independent?

Solution : H.W

$Gamma(\alpha, \lambda): M(t) = \left(\frac{\lambda}{\lambda-t}\right)^{\alpha}$; $Uniform(a,b): M(t) = \frac{e^{tb}-e^{ta}}{t(b-a)}$

$$M_X(t) = \left(\frac{3}{3-t}\right)^2; \qquad M_Y(t) = \frac{e^{2t}-1}{2t}\ ; \qquad M_Z(t) = \left(\frac{3}{3-t}\right)^5$$

$$\boldsymbol{M_{X+Y+Z}(t) = M_X(t)\ M_Y(t) M_Z(t)} = \left(\frac{\mathbf{3}}{\mathbf{3-t}}\right)^2 \left(\frac{e^{2t}-1}{2t}\right)\left(\frac{3}{3-t}\right)^5$$

$$\boldsymbol{M_{X+Y+Z}(t)} = \left(\frac{\mathbf{3}}{\mathbf{3-t}}\right)^7 \frac{e^{2t}-1}{2t}$$

Q14) If $X \sim Normal(2,3)$ independent of $Y \sim Normal(5,1)$, and $Z \sim Normal(20,21)$, with X, Y and Z independent, find $P(X+Y+Z < 25)$.

Solution :

$$X_i \sim Normal(\mu_i, \sigma_i^2) \Rightarrow \quad M_{X_i}(t) = e^{\mu_i t + \frac{1}{2}\sigma_i^2 t^2}$$

$$\boldsymbol{M_{X+Y+Z}(t) = M_X(t)\ M_Y(t) M_Z(t)}$$

$$= e^{\mu_1 t + \frac{1}{2}\sigma_1^2 t^2}\quad e^{\mu_2 t + \frac{1}{2}\sigma_2^2 t^2}\quad e^{\mu_3 t + \frac{1}{2}\sigma_3^2 t^2}$$

$$\boldsymbol{M_{X+Y+Z}(t)} = e^{t\sum_{i=1}^{3}\mu_i + \frac{1}{2}t^2\sum_{i=1}^{3}\sigma_i^2}$$

$$W = (X + Y + Z) \sim Normal\left(\sum_{i=1}^{3} \mu_i , \sum_{i=1}^{3} \sigma_i^2\right)$$

$$\sum_{i=1}^{3} \mu_i = 2 + 5 + 20 = 27 \;\; ; \qquad \sum_{i=1}^{3} \sigma_i^2 = 3 + 1 + 21 = 25$$

$$W \sim Normal\,(27\,, 25)$$

$$P(W < 25) = P\left(\frac{W - \mu}{\sigma} < \frac{25 - 27}{\sqrt{25}}\right) = P(Z < -0.4) = P(Z > 0.4) = 1 - P(Z < 0.4)$$

$$= 1 - 0.6554 = 0.3446$$