Eigenvalues and Eigenvectors

- Eigenvalues and Eigenvectors
- Diagonalization

Eigenvalues and Eigenvectors

- Eigenvalue problem (one of the most important problems in the linear algebra):

  If A is an $n \times n$ matrix, do there exist nonzero vectors $\mathbf{x}$ in R^n

  such that $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$?

  (The term eigenvalue is from the German word *Eigenwert*, meaning "proper value")

- Eigenvalue and Eigenvector:

  A: an $n \times n$ matrix

  λ: a scalar (could be **zero**)

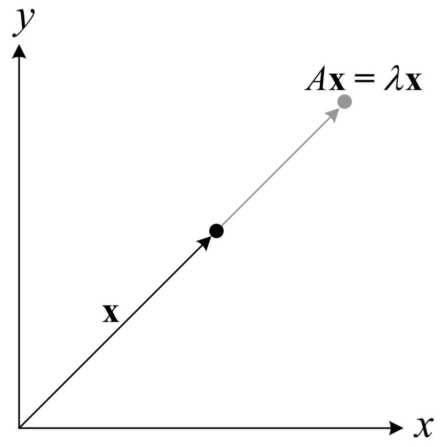
  $\mathbf{x}$: a **nonzero** vector in R^n

Eigenvalue

$$A\underset{\uparrow}{\mathbf{x}} = \overset{\downarrow}{\lambda}\underset{\uparrow}{\mathbf{x}}$$

Eigenvector

※ Geometric Interpretation



Example: Verifying eigenvalues and eigenvectors

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \quad \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Eigenvalue

$$A\mathbf{x}_1 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2\mathbf{x}_1$$

Eigenvector

※ In fact, for each eigenvalue, it has infinitely many eigenvectors. For $\lambda = 2$, $[3\ 0]^T$ or $[5\ 0]^T$ are both corresponding eigenvectors. Moreover, $([3\ 0] + [5\ 0])^T$ is still an eigenvector.

Eigenvalue

$$A\mathbf{x}_2 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -1\begin{bmatrix} 0 \\ 1 \end{bmatrix} = (-1)\mathbf{x}_2$$

Eigenvector

Theorem: The eigenspace corresponding to λ of matrix A

If A is an $n\times n$ matrix with an eigenvalue λ, then the set of all eigenvectors of λ **together with the zero vector** is a subspace of R^n. This subspace is called the eigenspace of λ.

Proof:

(1) Note 0 is in the set. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ are eigenvectors corresponding to λ

$$\text{(i.e., } A\mathbf{x}_1 = \lambda\mathbf{x}_1,\ \ A\mathbf{x}_2 = \lambda\mathbf{x}_2\text{)}$$

(2) $A(\mathbf{x}_1+\mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \lambda\mathbf{x}_1 + \lambda\mathbf{x}_2 = \lambda(\mathbf{x}_1+\mathbf{x}_2)$

(i.e., $\mathbf{x}_1+\mathbf{x}_2$ is also an eigenvector corresponding to λ)

(3) $A(c\mathbf{x}_1) = c(A\mathbf{x}_1) = c(\lambda\mathbf{x}_1) = \lambda(c\mathbf{x}_1)$

(i.e., $c\mathbf{x}_1$ is also an eigenvector corresponding to λ)

Since this set is closed under vector addition and scalar multiplication, this set is a subspace of R^n

Example: Examples of eigenspaces on the *xy*-plane

For the matrix A as follows, the corresponding eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 1$:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution:

For the eigenvalue $\lambda_1 = -1$, corresponding vectors are any vectors on the x-axis

$$A\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} -x \\ 0 \end{bmatrix} = \boxed{-1}\begin{bmatrix} x \\ 0 \end{bmatrix}$$

※ Thus, the eigenspace corresponding to $\lambda = -1$ is the x-axis, which is a subspace of R^2

For the eigenvalue $\lambda_2 = 1$, corresponding vectors are any vectors on the y-axis

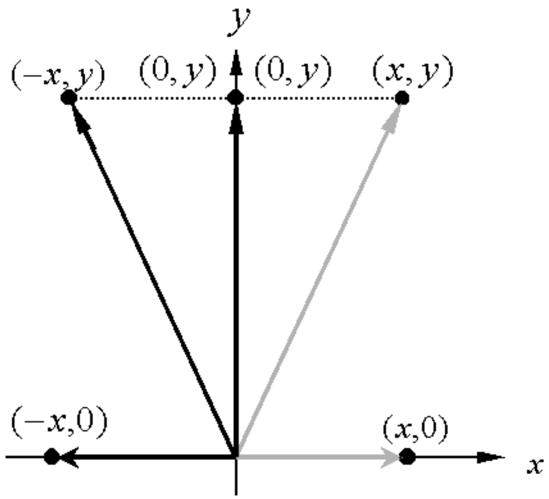
$$A\begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix} = \boxed{1}\begin{bmatrix} 0 \\ y \end{bmatrix}$$

※ Thus, the eigenspace corresponding to $\lambda = 1$ is the y-axis, which is a subspace of R^2

※ Geometrically speaking, multiplying a vector (x, y) in R^2 by the matrix A corresponds to a reflection to the y-axis, i.e., left multiplying A to **v** can transform **v** to another vector in the same vector space

$$A\mathbf{v} = A\begin{bmatrix} x \\ y \end{bmatrix} = A\left(\begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \end{bmatrix}\right) = A\begin{bmatrix} x \\ 0 \end{bmatrix} + A\begin{bmatrix} 0 \\ y \end{bmatrix}$$

$$= -1\begin{bmatrix} x \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$



Theorem: Finding eigenvalues and eigenvectors of a matrix $A \in M_{n\times n}$

Let A be an $n\times n$ matrix.

(1) An eigenvalue of A is a scalar λ such that $\det(\lambda I - A) = 0$

(2) The eigenvectors of A corresponding to λ are the nonzero solutions of $(\lambda I - A)\mathbf{x} = \mathbf{0}$

Note: follwing the definition of the eigenvalue problem

$A\mathbf{x} = \lambda\mathbf{x} \;\Rightarrow\; A\mathbf{x} = \lambda I\mathbf{x} \;\Rightarrow\; (\lambda I - A)\mathbf{x} = \mathbf{0}$ (homogeneous system)

$(\lambda I - A)\mathbf{x} = \mathbf{0}$ has nonzero solutions for $\mathbf{x}$ iff $\det(\lambda I - A) = 0$

- Characteristic equation of A:

$$\det(\lambda I - A) = 0$$

- Characteristic polynomial of $A \in M_{n\times n}$:

$$\det(\lambda I - A) = \left|(\lambda I - A)\right| = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

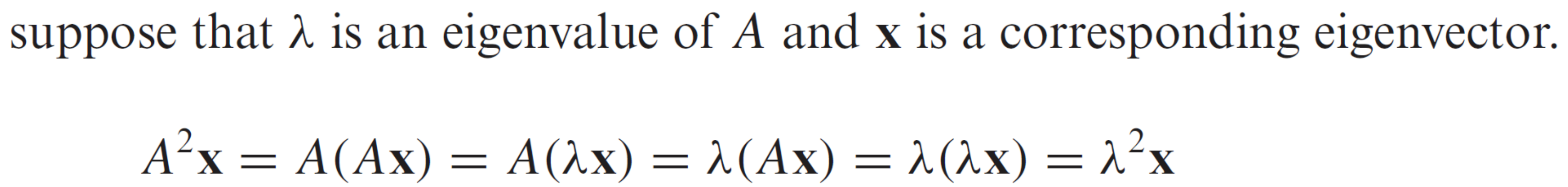
Theorem: Eigenvalues and Invertibility

A square matrix A is invertible if and only if $\lambda = 0$ is not an eigenvalue.

Theorem: Eigenvalues of Powers of a Matrix

If k is a positive integer and λ is an eigenvalue of a matrix A, and x is a corresponding eigenvector, then λ^k is an eigenvalue of A and x is a corresponding eigenvector.

Proof: For $k = 2$,



suppose that λ is an eigenvalue of A and $\mathbf{x}$ is a corresponding eigenvector.

$$A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda(A\mathbf{x}) = \lambda(\lambda\mathbf{x}) = \lambda^2\mathbf{x}$$

Example: Finding eigenvalues and eigenvectors

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$$

Solution: Characteristic equation:

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & 12 \\ -1 & \lambda + 5 \end{vmatrix}$$

$$= \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0$$

$$\Rightarrow \lambda = -1, -2$$

Eigenvalue: $\lambda_1 = -1, \lambda_2 = -2$

$$(1)\ \lambda_1 = -1 \ \Rightarrow (\lambda_1 I - A)\mathbf{x} = \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \xrightarrow{\text{G.-J. E.}} \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix} = t \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad t \neq 0$$

$$(2)\ \lambda_2 = -2 \ \Rightarrow (\lambda_2 I - A)\mathbf{x} = \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \xrightarrow{\text{G.-J. E.}} \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3s \\ s \end{bmatrix} = s \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad s \neq 0$$

Example: Finding eigenvalues and eigenvectors

Find the eigenvalues and corresponding eigenvectors for the matrix A. What is the dimension of the eigenspace of each eigenvalue?

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Solution: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^3 = 0$$

Eigenvalue: $\lambda = 2$

The eigenspace of $\lambda = 2$:

$$(\lambda I - A)\mathbf{x} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \neq 0$$

$$\left\{ s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \middle| \, s, t \in R \right\}$$: the eigenspace of A corresponding to $\lambda = 2$

Thus, the dimension of its eigenspace is 2

Notes:

(1) If an eigenvalue λ_1 occurs as a multiple root (k times) for the characteristic polynomial, then λ_1 has multiplicity k

(2) The multiplicity of an eigenvalue is greater than or equal to the dimension of its eigenspace. (In Ex, k is 3 and the dimension of its eigenspace is 2)

Example: Finding Eigenvalues and Eigenvectors

Find the eigenvalues of the matrix A and find a basis for each of the corresponding eigenspaces

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

Solution: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda-1 & 0 & 0 & 0 \\ 0 & \lambda-1 & -5 & 10 \\ -1 & 0 & \lambda-2 & 0 \\ -1 & 0 & 0 & \lambda-3 \end{vmatrix}$$

$$= (\lambda-1)^2(\lambda-2)(\lambda-3) = 0$$

Eigenvalues: $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$

※ According to the previous slide, the dimension of the eigenspace of $\lambda_1 = 1$ is at most to be 2

※ For $\lambda_2 = 2$ and $\lambda_3 = 3$, the dimensions of their eigenspaces are at most to be 1

(1) $\lambda_1 = 1 \quad \Rightarrow (\lambda_1 I - A)\mathbf{x} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 10 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$\overset{\text{G.-J.E.}}{\Rightarrow} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2t \\ s \\ 2t \\ t \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad s, t \neq 0$$

$\Rightarrow \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace corresponding to $\lambda_1 = 1$

※The dimension of the eigenspace of $\lambda_1 = 1$ is 2

$$(2)\ \lambda_2 = 2 \ \Rightarrow (\lambda_2 I - A)\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 10 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\overset{\text{G.-J.E.}}{\Rightarrow} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5t \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \quad t \neq 0$$

$\Rightarrow \left\{ \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for the eigenspace corresponding to $\lambda_2 = 2$

※The dimension of the eigenspace of $\lambda_2 = 2$ is 1

$$(3)\ \lambda_3 = 3 \ \Rightarrow (\lambda_3 I - A)\mathbf{x} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & -5 & 10 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\overset{\text{G.-J.E.}}{\Rightarrow} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -5t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}, \quad t \neq 0$$

$$\Rightarrow \left\{ \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$$ is a basis for the eigenspace corresponding to $\lambda_3 = 3$

※The dimension of the eigenspace of $\lambda_3 = 3$ is 1

Theorem: Eigenvalues for triangular matrices

If A is an $n \times n$ triangular matrix, then its eigenvalues are the entries on its main diagonal

Example: Finding eigenvalues for triangular and diagonal matrices

$$\text{(a) } A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 3 & -3 \end{bmatrix} \qquad \text{(b) } A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Solution:

$$\text{(a) } |\lambda I - A| = \begin{vmatrix} \lambda - 2 & 0 & 0 \\ 1 & \lambda - 1 & 0 \\ -5 & -3 & \lambda + 3 \end{vmatrix} = (\lambda - 2)(\lambda - 1)(\lambda + 3) = 0$$

$$\Rightarrow \lambda_1 = 2,\ \lambda_2 = 1,\ \lambda_3 = -3$$

$$\text{(b) } \lambda_1 = -1,\ \lambda_2 = 2,\ \lambda_3 = 0,\ \lambda_4 = -4,\ \lambda_5 = 3$$

※According to Thm., the determinant of a triangular matrix is the product of the entries on the main diagonal

Example: Finding eigenvalues and eigenvectors for standard matrices

Find the eigenvalues and corresponding eigenvectors for

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Solution:

$$|\lambda I - A| = \begin{bmatrix} \lambda - 1 & -3 & 0 \\ -3 & \lambda - 1 & 0 \\ 0 & 0 & \lambda + 2 \end{bmatrix} = (\lambda + 2)^2 (\lambda - 4) = 0$$

$\Rightarrow$ eigenvalues $\lambda_1 = 4,\ \lambda_2 = -2$

For $\lambda_1 = 4$, the corresponding eigenvector is $(1,\ 1,\ 0)$.

For $\lambda_2 = -2$, the corresponding eigenvectors are $(1,\ -1,\ 0)$
and $(0,\ 0,\ 1)$.

Diagonalization

- Diagonalization problem:

  For a square matrix A, does there exist an invertible matrix P such that $P^{-1}AP$ is diagonal?

- Diagonalizable matrix:

  Definition 1: A square matrix A is called **diagonalizable** if there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix (i.e., P diagonalizes A)

  Definition 2: A square matrix A is called **diagonalizable** if A is **similar** to a diagonal matrix

  ※ Remember two square matrices A and B are **similar** if there exists an invertible matrix P such that $B = P^{-1}AP$.

Notes:

This section shows that the eigenvalue and eigenvector problem is closely related to the diagonalization problem

Theorem: Similar matrices have the same eigenvalues

If A and B are similar $n \times n$ matrices, then they have the same eigenvalues

Proof:

A and B are similar $\Rightarrow B = P^{-1}AP$

For any diagonal matrix in the form of $D = \lambda I$, $P^{-1}DP = D$

Consider the characteristic equation of B:

$$\begin{aligned}
|\lambda I - B| &= |\lambda I - P^{-1}AP| = |P^{-1}\lambda IP - P^{-1}AP| = |P^{-1}(\lambda I - A)P| \\
&= |P^{-1}||\lambda I - A||P| = |P^{-1}||P||\lambda I - A| = |P^{-1}P||\lambda I - A| \\
&= |\lambda I - A|
\end{aligned}$$

Since A and B have the same characteristic equation, they are with the same eigenvalues

※ Note that the eigenvectors of A and B are not necessarily identical

Example: Eigenvalue problems and diagonalization programs

$$A=\begin{bmatrix}1 & 3 & 0\\ 3 & 1 & 0\\ 0 & 0 & -2\end{bmatrix}$$

Solution: Characteristic equation:

$$|\lambda I-A|=\begin{vmatrix}\lambda-1 & -3 & 0\\ -3 & \lambda-1 & 0\\ 0 & 0 & \lambda+2\end{vmatrix}=(\lambda-4)(\lambda+2)^2=0$$

$$\text{The eigenvalues}: \lambda_1=4,\ \lambda_2=-2,\ \lambda_3=-2$$

$$(1)\ \lambda=4 \Rightarrow \text{ the eigenvector } \mathbf{p}_1=\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$$

$$(2)\ \lambda = -2 \Rightarrow \text{the eigenvector } \mathbf{p}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},\quad \mathbf{p}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- Note: If $P = [\mathbf{p}_2 \quad \mathbf{p}_1 \quad \mathbf{p}_3]$

$$= \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \Rightarrow P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

※ The above example can verify Thm. since the eigenvalues for both A and $P^{-1}AP$ are the same to be 4, −2, and −2

※ The reason why the matrix P is constructed with the eigenvectors of A is demonstrated in next Thm. on the next slide

Theorem: Condition for diagonalization

An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.

※ If there are n linearly independent eigenvectors, it does not imply that there are n distinct eigenvalues. It is possible to have only one eigenvalue with the multiplicity n, and there are n linearly independent eigenvectors

※ On the other hand, if there are n distinct eigenvalues, then there are n linearly independent eigenvectors, and thus A must be diagonalizable

Proof: $(\Rightarrow)$

Since A is diagonalizable, there exists an invertible P s.t. $D = P^{-1}AP$ is diagonal. Let $P = [\mathbf{p}_1 \ \mathbf{p}_2 \ \cdots \ \mathbf{p}_n]$ and $D = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$, then

$$PD = [\mathbf{p}_1 \ \mathbf{p}_2 \ \cdots \ \mathbf{p}_n]\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$= [\lambda_1\mathbf{p}_1 \ \lambda_2\mathbf{p}_2 \ \cdots \ \lambda_n\mathbf{p}_n]$$

$\because AP = PD$ (since $D = P^{-1}AP$)

$\therefore [A\mathbf{p}_1 \; A\mathbf{p}_2 \; \cdots \; A\mathbf{p}_n] = [\lambda_1\mathbf{p}_1 \; \lambda_2\mathbf{p}_2 \; \cdots \; \lambda_n\mathbf{p}_n]$

$\Rightarrow A\mathbf{p}_i = \lambda_i\mathbf{p}_i, \;\; i = 1, \; 2, \ldots, \; n$

(The above equations imply the column vectors $\mathbf{p}_i$ of P are eigenvectors of A, and the diagonal entries λ_i in D are eigenvalues of A)

Because A is diagonalizable $\Rightarrow P$ is invertible

$\Rightarrow$ Columns in P, i.e., $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n$, are linearly independent

(see Slide 4.101 in the lecture note)

Thus, A has n linearly independent eigenvectors

$(\Leftarrow)$

Since A has n linearly independent eigenvectors $\mathbf{p}_1, \mathbf{p}_2, \cdots \mathbf{p}_n$ with corresponding eigenvalues $\lambda_1, \lambda_2, \cdots \lambda_n$ (could be the same), then

$\Rightarrow A\mathbf{p}_i = \lambda_i\mathbf{p}_i, \;\; i = 1, \; 2, \ldots, \; n$

Let $P = [\mathbf{p}_1 \; \mathbf{p}_2 \; \cdots \; \mathbf{p}_n]$

$$
\begin{aligned}
AP &= A[\mathbf{p}_1\ \mathbf{p}_2\ \cdots\ \mathbf{p}_n] = [A\mathbf{p}_1\ A\mathbf{p}_2\ \cdots\ A\mathbf{p}_n] \\
&= [\lambda_1\mathbf{p}_1\ \lambda_2\mathbf{p}_2\ \cdots\ \lambda_n\mathbf{p}_n] \\
&= [\mathbf{p}_1\ \mathbf{p}_2\ \cdots\ \mathbf{p}_n]\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = PD
\end{aligned}
$$

Since $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n$ are linearly independent

$\Rightarrow$ P is invertible

$\because AP = PD \therefore P^{-1}AP = D$

$\Rightarrow$ A is diagonalizable

(according to the definition of the diagonalizable matrix)

※ Note that $\mathbf{p}_i$'s are linearly independent eigenvectors and the diagonal entries λ_i in the resulting diagonalized D are eigenvalues of A

Example: A matrix that is not diagonalizable

Show that the following matrix is not diagonalizable

$$A=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$$

Solution: Characteristic equation:

$$|\lambda I-A|=\begin{vmatrix}\lambda-1 & -2\\ 0 & \lambda-1\end{vmatrix}=(\lambda-1)^2=0$$

The eigenvalue $\lambda_1=1,$ and then solve $(\lambda_1 I-A)\mathbf{x}=\mathbf{0}$ for eigenvectors

$$\lambda_1 I-A=I-A=\begin{bmatrix}0 & -2\\ 0 & 0\end{bmatrix}\Rightarrow \text{eigenvector } \mathbf{p}_1=\begin{bmatrix}1\\ 0\end{bmatrix}$$

Since A does not have two linearly independent eigenvectors,

A is not diagonalizable

- Steps for diagonalizing an $n \times n$ square matrix:

Step 1: Find n linearly independent eigenvectors $\mathbf{p}_1, \mathbf{p}_2, \cdots \mathbf{p}_n$

for A with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$

Step 2: Let $P = [\mathbf{p}_1 \ \mathbf{p}_2 \ \cdots \ \mathbf{p}_n]$

Step 3:

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

where $A\mathbf{p}_i = \lambda_i \mathbf{p}_i, \quad i = 1, \ 2, \ldots, \ n$

Example: Diagonalizing a matrix

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

Find a matrix P such that $P^{-1}AP$ is diagonal.

Solution: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & 1 & 1 \\ -1 & \lambda - 3 & -1 \\ 3 & -1 & \lambda + 1 \end{vmatrix} = (\lambda - 2)(\lambda + 2)(\lambda - 3) = 0$$

The eigenvalues : $\lambda_1 = 2,\ \lambda_2 = -2,\ \lambda_3 = 3$

$$\lambda_1 = 2 \Rightarrow \lambda_1 I - A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 3 & -1 & 3 \end{bmatrix} \xrightarrow{\text{G.-J. E.}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} \quad \Rightarrow \quad \text{eigenvector } \mathbf{p}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -2 \Rightarrow \lambda_2 I - A = \begin{bmatrix} -3 & 1 & 1 \\ -1 & -5 & -1 \\ 3 & -1 & -1 \end{bmatrix} \xrightarrow{\text{G.-J. E.}} \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}t \\ -\frac{1}{4}t \\ t \end{bmatrix} \quad \Rightarrow \quad \text{eigenvector } \mathbf{p}_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$$\lambda_3 = 3 \Rightarrow \lambda_3 I - A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 3 & -1 & 4 \end{bmatrix} \xrightarrow{\text{G.-J. E.}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix} \quad \Rightarrow \quad \text{eigenvector } \mathbf{p}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3] = \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & 1 \end{bmatrix} \text{ and it follows that}$$

$$P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Note: a quick way to calculate A^k based on the diagonalization technique

$$(1)\ D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \Rightarrow D^k = \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{bmatrix}$$

$$(2)\ D = P^{-1}AP \Rightarrow D^k = \underbrace{\underbrace{P^{-1}AP}\ \underbrace{P^{-1}AP}\cdots\underbrace{P^{-1}AP}}_{\text{repeat } k \text{ times}} = P^{-1}A^kP$$

$$A^k = PD^kP^{-1}, \text{ where } D^k = \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{bmatrix}$$

Theorem: Sufficient conditions for diagonalization

If an $n \times n$ matrix A has n distinct eigenvalues, then the corresponding eigenvectors are linearly independent and thus A is diagonalizable according to last Thm.

Proof:

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be distinct eigenvalues and corresponding eigenvectors be $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$. In addition, consider that the first m eigenvectors are linearly independent, but the first m+1 eigenvectors are linearly dependent, i.e.,

$$\mathbf{x}_{m+1} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_m\mathbf{x}_m, \qquad (1)$$

where c_i's are not all zero. Multiplying both sides of Eq. (1) by A yields

$$\begin{aligned} A\mathbf{x}_{m+1} &= Ac_1\mathbf{x}_1 + Ac_2\mathbf{x}_2 + \cdots + Ac_m\mathbf{x}_m \\ \lambda_{m+1}\mathbf{x}_{m+1} &= c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 + \cdots + c_m\lambda_m\mathbf{x}_m \qquad (2) \end{aligned}$$

On the other hand, multiplying both sides of Eq. (1) by λ_{m+1} yields

$$\lambda_{m+1}\mathbf{x}_{m+1} = c_1\lambda_{m+1}\mathbf{x}_1 + c_2\lambda_{m+1}\mathbf{x}_2 + \cdots + c_m\lambda_{m+1}\mathbf{x}_m \qquad (3)$$

Now, subtracting Eq. (2) from Eq. (3) produces

$$c_1(\lambda_{m+1} - \lambda_1)\mathbf{x}_1 + c_2(\lambda_{m+1} - \lambda_2)\mathbf{x}_2 + \cdots + c_m(\lambda_{m+1} - \lambda_m)\mathbf{x}_m = \mathbf{0}$$

Since the first m eigenvectors are linearly independent, we can infer that all coefficients of this equation should be zero, i.e.,

$$c_1(\lambda_{m+1} - \lambda_1) = c_2(\lambda_{m+1} - \lambda_2) = \cdots = c_m(\lambda_{m+1} - \lambda_m) = 0$$

Because all the eigenvalues are distinct, it follows all c_i's equal to 0, which contradicts our assumption that $\mathbf{x}_{m+1}$ can be expressed as a linear combination of the first m eigenvectors. So, the set of n eigenvectors is linearly independent given n distinct eigenvalues, and according to previous Thm., we can conclude that A is diagonalizable

Example: Determining whether a matrix is diagonalizable

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

Solution: Because A is a triangular matrix, its eigenvalues are

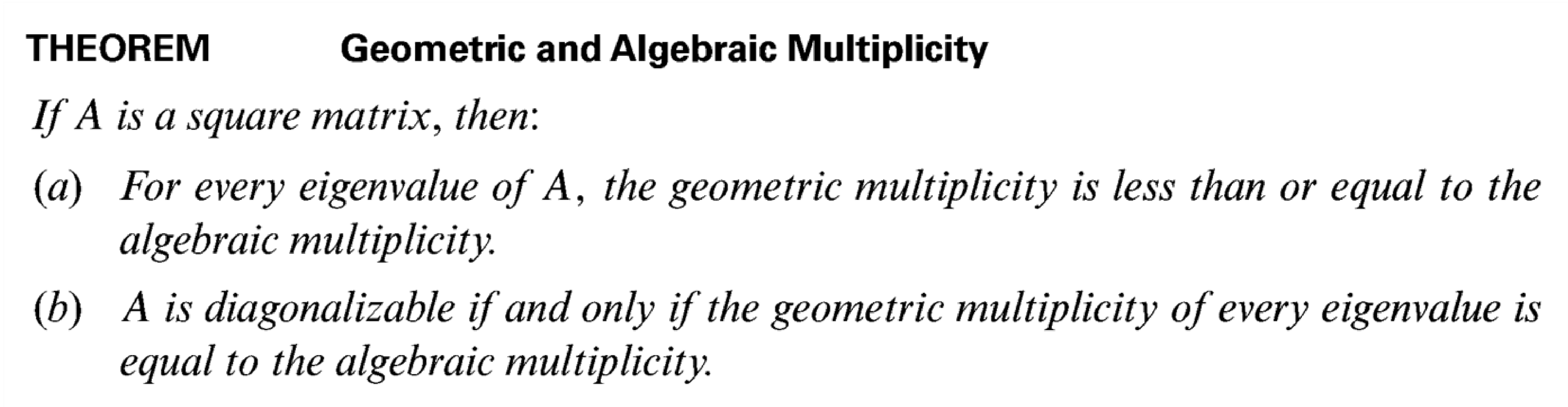
$$\lambda_1 = 1,\ \lambda_2 = 0,\ \lambda_3 = -3$$

According to Thm., because these three values are distinct, A is diagonalizable

For an eigenvalue λ_0: let E $_{\lambda 0}$ be the eigenspace corresponding to λ_0

Geometric multiplicity of λ_0 := dim(E $_{\lambda 0}$)

Algebraic multiplicity of λ_0 := number of times λ - λ_0 appears in the characteristic polynomial.



THEOREM **Geometric and Algebraic Multiplicity**

If A is a square matrix, then:

(a) *For every eigenvalue of A, the geometric multiplicity is less than or equal to the algebraic multiplicity.*

(b) *A is diagonalizable if and only if the geometric multiplicity of every eigenvalue is equal to the algebraic multiplicity.*

Example 3: Solve the eigenvalue problem A**x** = λ**x** and find the eigenspace, algebraic multiplicity, and geometric multiplicity for each eigenvalue.

$$A = \begin{bmatrix} -4 & -3 & 6 \\ 0 & -1 & 0 \\ -3 & -3 & 5 \end{bmatrix}$$

Step 1: Write down the characteristic equation of A and solve for its eigenvalues.

$$p(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda+4 & 3 & -6 \\ 0 & \lambda+1 & 0 \\ 3 & 3 & \lambda-5 \end{vmatrix} = (-1)^4(\lambda+1)\begin{vmatrix} \lambda+4 & -6 \\ 3 & \lambda-5 \end{vmatrix}$$

$$p(\lambda) = (\lambda+1)((\lambda+4)(\lambda-5)+18) = 0$$

$$= (\lambda+1)(\lambda^2-\lambda-2) = -(\lambda+1)(\lambda-2)(\lambda+1) = 0$$

$$= (\lambda-2)(\lambda+1)^2 = 0.$$

So the eigenvalues are $\lambda_1 = 2, \lambda_2 = -1.$

Since the factor (λ - 2) is first power, $\lambda_1 = 2$ is not a repeated root. $\lambda_1 = 2$ has an algebraic multiplicity of 1. On the other hand, the factor (λ +1) is squared, $\lambda_2 = -1$ is a repeated root, and it has an algebraic multiplicity of 2.

Step 2: Use Gaussian elimination with back-substitution to solve (λI - A) **x** = **0** for λ_1 and λ_2.

For $\lambda_1 = 2$, the augmented matrix for the system is

$$\left[2I - A \mid \vec{0}\right] = \begin{bmatrix} 6 & 3 & -6 & 0 \\ 0 & 3 & 0 & 0 \\ 3 & 3 & -3 & 0 \end{bmatrix} \sim \begin{matrix} \frac{1}{6}r1 \to r1 \\ \frac{1}{3}r2 \to r2 \\ r3 \end{matrix} \begin{bmatrix} 1 & 1/2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 3 & -3 & 0 \end{bmatrix}$$

$$\sim \begin{matrix} r1 \\ r2 \\ -3r1 + r3 \to r3 \end{matrix} \begin{bmatrix} 1 & 1/2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3/2 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{matrix} r1 \\ r2 \\ -\frac{3}{2}r2 + r3 \to r3 \end{matrix} \begin{bmatrix} 1 & 1/2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In this case,

$x_3 = r$, $x_2 = 0$, and

$x_1 = -1/2(0) + r$

$= 0 + r = r.$

Thus, the eigenvector corresponding to $\lambda_1 = 2$ is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} r \\ 0 \\ r \end{bmatrix} = r\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, r \neq 0. \textit{ If we choose } \vec{p}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

$$\textit{then } B_1 = \left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\} \textit{is a basis for the eigenspace of } \lambda_1 = 2.$$

$$E_{\lambda_1} = span(\{\vec{p}_1\}) \textit{ and } \dim(E_{\lambda_1}) = 1, \textit{ so the geometric multiplicity is } 1.$$

$$A\vec{x} = 2\vec{x} \textit{ or } (2I - A)\vec{x} = \vec{0}.$$

$$\begin{bmatrix} -4 & -3 & 6 \\ 0 & -1 & 0 \\ -3 & -3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -4+6 \\ 0 \\ -3+5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

For λ_2 = -1, the augmented matrix for the system is

$$\left[(-1)I - A \mid \vec{0}\right] = \begin{bmatrix} 3 & 3 & -6 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 3 & -6 & 0 \end{bmatrix} \sim \begin{matrix} \frac{1}{3}r1 \to r1 \\ r2 \\ r3 \end{matrix} \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 3 & -6 & 0 \end{bmatrix}$$

$$\sim \begin{matrix} r1 \\ r2 \\ -3r1 + r3 \to r3 \end{matrix} \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3 = t, x_2 = s, and x_1 = -s + 2t. Thus, the solution has two linearly independent eigenvectors for λ_2 = -1 with

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s + 2t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, s \neq 0, t \neq 0.$$

$$\text{If we choose } \vec{p}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \vec{p}_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \text{ then } B_2 = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\text{is a basis for } E_{\lambda_2} = span(\{\vec{p}_2, \vec{p}_3\}) \text{ and } \dim(E_{\lambda_2}) = 2,$$

$$\text{so the geometric multiplicity is } 2.$$

Since the geometric multiplicity is equal to the algebraic multiplicity for each distinct eigenvalue, we found three linearly independent eigenvectors. The matrix A is diagonalizable since P = [$\mathbf{p}_1$ $\mathbf{p}_2$ $\mathbf{p}_3$] is nonsingular.