1.6 More on Linear Systems and Invertible Matrices

Number of Solutions of a Linear System

THEOREM:

A system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.

Solving Linear Systems by Matrix Inversion

THEOREM:

If A is an invertible n × n matrix, then for each n×1 matrix **b**, the system of equations $\boldsymbol{Ax} = \boldsymbol{b}$ has exactly one solution, namely, $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$.

EXAMPLE 1:

Solution of a Linear System Using A^{-1}, Consider the system of linear equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5 \\ 2x_1 + 5x_2 + 3x_3 &= 3 \\ x_1 \qquad\quad + 8x_3 &= 17 \end{aligned}$$

EXAMPLE 2:

Solve the systems

(a)
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 2x_1 + 5x_2 + 3x_3 &= 5 \\ x_1 \qquad\quad + 8x_3 &= 9 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 5x_2 + 3x_3 &= 6 \\ x_1 \qquad\quad + 8x_3 &= -6 \end{aligned}$$

Properties of Invertible Matrices

THEOREM:

Let A be a square matrix.

(a) If B is a square matrix satisfying $BA = I$, then $B = A^{-1}$.
(b) If B is a square matrix satisfying $AB = I$, then $B = A^{-1}$.

Proof (a) Assume that $BA = I$. If we can show that A is invertible, the proof can be completed by multiplying $BA = I$ on both sides by A^{-1} to obtain

$$BAA^{-1} = IA^{-1} \quad \text{or} \quad BI = IA^{-1} \quad \text{or} \quad B = A^{-1}$$

Equivalence Theorem

THEOREM:

If **A** is an n × n matrix, then the following are equivalent.

(a) **A** is invertible.

(b) $Ax = 0$ has only the trivial solution.

(c) The reduced row echelon form of **A** is I_n.

(d) **A** is expressible as a product of elementary matrices.

(e) $Ax = b$ is consistent for every n×1 matrix **b**.

(f) $Ax = b$ has exactly one solution for every n×1 matrix **b**.

THEOREM:

Let **A** and **B** be square matrices of the same size. If **AB** is **invertible**, then **A** and **B** must also be **invertible**.

A Fundamental Problem

Let **A** be a fixed m × n matrix. Find all m×1 matrices **b** such that the system of equations $Ax = b$ is consistent.

EXAMPLE 3:

What conditions must b_1, b_2, and b_3 satisfy in order for the system of equations

$$\begin{aligned} x_1 + x_2 + 2x_3 &= b_1 \\ x_1 \qquad + x_3 &= b_2 \\ 2x_1 + x_2 + 3x_3 &= b_3 \end{aligned}$$

to be consistent?

EXAMPLE 4:

What conditions must b_1, b_2, and b_3 satisfy in order for the system of equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= b_1 \\ 2x_1 + 5x_2 + 3x_3 &= b_2 \\ x_1 \qquad\quad + 8x_3 &= b_3 \end{aligned}$$

to be consistent?

Exercise Set 1.6

► In Exercises 1–8, solve the system by inverting the coefficient matrix and using Theorem 1.6.2. ◄

1. $$\begin{aligned} x_1 + x_2 &= 2 \\ 5x_1 + 6x_2 &= 9 \end{aligned}$$

2. $$\begin{aligned} 4x_1 - 3x_2 &= -3 \\ 2x_1 - 5x_2 &= 9 \end{aligned}$$

3. $$\begin{aligned} x_1 + 3x_2 + x_3 &= 4 \\ 2x_1 + 2x_2 + x_3 &= -1 \\ 2x_1 + 3x_2 + x_3 &= 3 \end{aligned}$$

4. $$\begin{aligned} 5x_1 + 3x_2 + 2x_3 &= 4 \\ 3x_1 + 3x_2 + 2x_3 &= 2 \\ x_2 + x_3 &= 5 \end{aligned}$$

5. $$\begin{aligned} x + y + z &= 5 \\ x + y - 4z &= 10 \\ -4x + y + z &= 0 \end{aligned}$$

6. $$\begin{aligned} - x - 2y - 3z &= 0 \\ w + x + 4y + 4z &= 7 \\ w + 3x + 7y + 9z &= 4 \\ -w - 2x - 4y - 6z &= 6 \end{aligned}$$

7. $$\begin{aligned} 3x_1 + 5x_2 &= b_1 \\ x_1 + 2x_2 &= b_2 \end{aligned}$$

8. $$\begin{aligned} x_1 + 2x_2 + 3x_3 &= b_1 \\ 2x_1 + 5x_2 + 5x_3 &= b_2 \\ 3x_1 + 5x_2 + 8x_3 &= b_3 \end{aligned}$$

► In Exercises 9–12, solve the linear systems together by reducing the appropriate augmented matrix. ◄

9. $$\begin{aligned} x_1 - 5x_2 &= b_1 \\ 3x_1 + 2x_2 &= b_2 \end{aligned}$$
(i) $b_1 = 1, \quad b_2 = 4$ (ii) $b_1 = -2, \quad b_2 = 5$

10. $$\begin{aligned} -x_1 + 4x_2 + x_3 &= b_1 \\ x_1 + 9x_2 - 2x_3 &= b_2 \\ 6x_1 + 4x_2 - 8x_3 &= b_3 \end{aligned}$$
(i) $b_1 = 0, \quad b_2 = 1, \quad b_3 = 0$
(ii) $b_1 = -3, \quad b_2 = 4, \quad b_3 = -5$

11. $$\begin{aligned} 4x_1 - 7x_2 &= b_1 \\ x_1 + 2x_2 &= b_2 \end{aligned}$$
(i) $b_1 = 0, \quad b_2 = 1$ (ii) $b_1 = -4, \quad b_2 = 6$
(iii) $b_1 = -1, \quad b_2 = 3$ (iv) $b_1 = -5, \quad b_2 = 1$

12. $$\begin{aligned} x_1 + 3x_2 + 5x_3 &= b_1 \\ -x_1 - 2x_2 &= b_2 \\ 2x_1 + 5x_2 + 4x_3 &= b_3 \end{aligned}$$
(i) $b_1 = 1, \quad b_2 = 0, \quad b_3 = -1$
(ii) $b_1 = 0, \quad b_2 = 1, \quad b_3 = 1$
(iii) $b_1 = -1, \quad b_2 = -1, \quad b_3 = 0$

► In Exercises 13–17, determine conditions on the b_i's, if any, in order to guarantee that the linear system is consistent. ◄

13. $$\begin{aligned} x_1 + 3x_2 &= b_1 \\ -2x_1 + x_2 &= b_2 \end{aligned}$$

14. $$\begin{aligned} 6x_1 - 4x_2 &= b_1 \\ 3x_1 - 2x_2 &= b_2 \end{aligned}$$

15. $$\begin{aligned} x_1 - 2x_2 + 5x_3 &= b_1 \\ 4x_1 - 5x_2 + 8x_3 &= b_2 \\ -3x_1 + 3x_2 - 3x_3 &= b_3 \end{aligned}$$

16. $$\begin{aligned} x_1 - 2x_2 - x_3 &= b_1 \\ -4x_1 + 5x_2 + 2x_3 &= b_2 \\ -4x_1 + 7x_2 + 4x_3 &= b_3 \end{aligned}$$

17. $$\begin{aligned} x_1 - x_2 + 3x_3 + 2x_4 &= b_1 \\ -2x_1 + x_2 + 5x_3 + x_4 &= b_2 \\ -3x_1 + 2x_2 + 2x_3 - x_4 &= b_3 \\ 4x_1 - 3x_2 + x_3 + 3x_4 &= b_4 \end{aligned}$$

18. Consider the matrices
$$A = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & -2 \\ 3 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$
(a) Show that the equation $A\mathbf{x} = \mathbf{x}$ can be rewritten as $(A - I)\mathbf{x} = \mathbf{0}$ and use this result to solve $A\mathbf{x} = \mathbf{x}$ for $\mathbf{x}$.
(b) Solve $A\mathbf{x} = 4\mathbf{x}$.

► In Exercises 19–20, solve the matrix equation for X. ◄

19. $$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix} X = \begin{bmatrix} 2 & -1 & 5 & 7 & 8 \\ 4 & 0 & -3 & 0 & 1 \\ 3 & 5 & -7 & 2 & 1 \end{bmatrix}$$