1.5 Elementary Matrices and a Method for Finding A^{-1}

Definition:

Matrices **A** and **B** are said to be **row equivalent** if either (hence each) can be obtained from the other by a sequence of elementary row operations.

Definition:

A matrix **E** is called **an elementary matrix** if it can be obtained from an identity matrix by performing a single elementary row operation.

Example 1:

Elementary Matrices and Row Operations

$$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$$

Multiply the second row of I2 by−3.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Interchange the second and fourth rows of I_4 .

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Multiply the first row of I_3 by 1.

Row Operations by Matrix Multiplication

THEOREM

If the elementary matrix **E** results from performing a certain row operation on $\boldsymbol{I_m}$ and if **A** is an **m** × **n matrix**, then the product **EA** is the matrix that results when this same row operation is performed on A.

Example2:

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}$$

and consider the elementary matrix

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

Which results from adding 3 times the first row of I_3 to the third row. The product EA is

$$EA = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 9 \end{bmatrix}$$

Example 3:

Row Operations and Inverse Row Operations

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Multiply the second row by 7.

Multiply the second row by $\frac{1}{7}$.

THEOREM

Every elementary matrix is invertible, and the inverse is also an elementary matrix.

Equivalent Statements

THEOREM

If A is an $n \times n$ matrix, then the following statements are equivalent, that is, all true or all false.

(a) *A is invertible.*

(b) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(c) *The reduced row echelon form of A is I_n.*

(d) *A is expressible as a product of elementary matrices.*

Inversion Algorithm

To find the inverse of an invertible matrix A, find a sequence of elementary row operations that reduces A to the identity and then perform that same sequence of operations on In to obtain A^{-1}.

Example 4:

Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

Example 5:

Showing That a Matrix Is Not Invertible

Consider the matrix

$$A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$$