1.4 Inverses Algebraic Properties of Matrices

Properties of Matrix Addition and Scalar Multiplication

THEOREM

Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid.

(a) $A + B = B + A$ [Commutative law for matrix addition]

(b) $A + (B + C) = (A + B) + C$ [Associative law for matrix addition]

(c) $A(BC) = (AB)C$ Associative law for matrix multiplication]

(d) $A(B + C) = AB + AC$ [Left distributive law]

(e) $(B + C)A = BA + CA$ [Right distributive law]

(f) $A(B - C) = AB - AC$

(g) $(B - C)A = BA - CA$

(h) $a(B + C) = aB + aC$

(i) $a(B - C) = aB - aC$

(j) $(a + b)C = aC + bC$

(k) $(a - b)C = aC - bC$

(l) $a(bC) = (ab)C$

(m) $a(BC) = (aB)C = B(aC)$

Example 1:

As an illustration of the associative law for matrix multiplication, (AB)C =A(BC)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

Inverse of Matrices

Definition

If A is ***a square matrix***, and if a matrix B ***of the same size*** can be found such that ***AB=BA=I***, then A is said to be ***invertible*** (**or nonsingular**), and B is called an ***inverse*** of A. If no such matrix B can be found, then A is said to be singular.

Example:

Let

$$A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \; and \; B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Thus, A and B are invertible, and each is an inverse of the other.

Properties of Inverses

THEOREM

If **B** and **C** are both inverses of the matrix **A** then **B** = **C**.

determinant of the matrix

Definition

the determinant is a value that can be computed from the elements of a square matrix. The determinant of a matrix A is denoted det(A) or |A|.
In the case of a 2×2 matrix the determinant may be defined as:

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example :

$Find\ the\ \det(A)\ of\ matrix\ \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}?$

Calculating the Inverse of a 2x2 Matrix

Theorem

The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

is invertible if and only if ***ad-bc≠0***, in which case the inverse is giving by the formula

$$A^{-1} = \frac{1}{|A|} . \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \text{ or } A^{-1} = \frac{1}{ad-bc} . \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example:

In each part, determine whether the matrix is invertible. If so, find its inverse.

Solution:

$$(a)\ A = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix}, \qquad (b)\ A = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$$

(a) The determination of A is

det(A)=

(b) det(A)=

Properties of Inverses

THEOREM

If A and B are invertible matrices with the same size, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

EXAMPLE

Consider the matrices $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$

Powers of a Matrix

If A is a square matrix, then we define the **nonnegative integer powers** of A to be

$$A^0 = I \quad \text{and} \quad A^n = AA \cdots A \quad [n \text{ factors}]$$

and if A is **invertible**, then we define the negative integer powers of A to be

$$A^{-n} = (A^{-1})^n = A^{-1}A^{-1} \cdots A^{-1} \quad [n \text{ factors}]$$

Because these definitions parallel those for real numbers, the usual laws of nonnegative exponents hold; for example,

$$A^r A^s = A^{r+s} \quad \text{and} \quad (A^r)^s = A^{rs}$$

In addition, we have the following properties of negative exponents.

THEOREM

If A is **invertible** and **n** is a **nonnegative** integer, then :

(a) A^{-1} *is invertible and* $(A^{-1})^{-1} = A$.

(b) A^n *is invertible and* $(A^n)^{-1} = A^{-n} = (A^{-1})^n$.

(c) kA *is invertible for any nonzero scalar* k, *and* $(kA)^{-1} = k^{-1}A^{-1}$.

Matrix Polynomials

EXAMPLE

Find $p(A)$ for

$$p(x) = x^2 - 2x - 3 \quad \text{and} \quad A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}$$

Solution

$$\begin{aligned} p(A) &= A^2 - 2A - 3I \\ &= \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}^2 - 2\begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 4 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} -2 & 4 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

or more briefly, $p(A) = 0$. ◀

Properties of the Transpose

THEOREM

If the sizes of the matrices are such that the stated operations can be performed, then :

(a) $(A^T)^T = A$

(b) $(A + B)^T = A^T + B^T$

(c) $(A - B)^T = A^T - B^T$

(d) $(kA)^T = kA^T$

(e) $(AB)^T = B^T A^T$

THEOREM

If A is an ***invertible*** matrix, then A^T is also ***invertible*** and

$$(A^T)^{-1} = (A^{-1})^T$$

Proof We can establish the invertibility and obtain the formula at the same time by showing that

$$A^T(A^{-1})^T = (A^{-1})^T A^T = I$$

But from part (*e*) of Theorem 1.4.8 and the fact that $I^T = I$, we have

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$$

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$$

which completes the proof. ◀

▶ EXAMPLE 13 Inverse of a Transpose

Consider a general 2×2 invertible matrix and its transpose:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Since A is invertible, its determinant $ad - bc$ is nonzero. But the determinant of A^T is also $ad - bc$ (verify), so A^T is also invertible. It follows from Theorem 1.4.5 that

$$(A^T)^{-1} = \begin{bmatrix} \dfrac{d}{ad-bc} & -\dfrac{c}{ad-bc} \\ -\dfrac{b}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix}$$

which is the same matrix that results if A^{-1} is transposed (verify). Thus,

$$(A^T)^{-1} = (A^{-1})^T$$

as guaranteed by Theorem 1.4.9. ◀

Exercise Set 1.4

In Exercises 1–2, verify that the following matrices and scalars satisfy the stated properties of Theorem 1.4.1.

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 \\ 1 & -4 \end{bmatrix},$$

$$C = \begin{bmatrix} 4 & 1 \\ -3 & -2 \end{bmatrix}, \quad a = 4, \quad b = -7$$

1. (a) The associative law for matrix addition.

(b) The associative law for matrix multiplication.

(c) The left distributive law.

(d) $(a + b)C = aC + bC$

2. (a) $a(BC) = (aB)C = B(aC)$

(b) $A(B - C) = AB - AC$ (c) $(B + C)A = BA + CA$

(d) $a(bC) = (ab)C$

In Exercises 3–4, verify that the matrices and scalars in Exercise 1 satisfy the stated properties.

3. (a) $(A^T)^T = A$ (b) $(AB)^T = B^T A^T$

4. (a) $(A + B)^T = A^T + B^T$ (b) $(aC)^T = aC^T$

In Exercises 5–8, use Theorem 1.4.5 to compute the inverse of the matrix.

5. $A = \begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}$ **6.** $B = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$