1.2
Gaussian Elimination

Objectives:

- Define row echelon and reduced row echelon form.
- Applying elementary row operations to transform any matrix into echelon form and reduced echelon form.
- Using Row Reduction to Solve a Linear System

Definition:

A rectangular matrix is in **echelon** form (or **row echelon** form) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.

2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.

3. All entries in a column below a leading entry are zeros. If a matrix in echelon form satisfies the following additional conditions , then it is in **reduced echelon** form(or **reduced row echelon form**):

4. The leading entry in each nonzero row is 1.

5. Each leading 1 is the only nonzero entry in its column.

Echelon Form of a Matrix

1. Any rows of all zeros are below any other nonzero rows
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it
3. All entries in a column below a leading entry are zeros

$$\begin{bmatrix} 3 & 2 & 0 & 7 & 9 \\ 0 & 4 & 5 & 10 & 0 \\ 0 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced Echelon Form

1. Rows of zeros at the bottom
2. Leading entries go down and to the right
3. Zeros below leading entries
4. Every leading entry is 1
5. Zeros above and below leading entries

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 9 \\ 0 & 1 & 4 & 0 & -6 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 1:

Row Echelon and Reduced Row Echelon Form

- **The following matrices are in reduced row echelon form .**

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- **The following matrices are in row echelon form but not reduced row echelon form.**

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Row Echelon

A matrix in row echelon form has zeros below each leading 1

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced Row Echelon

A matrix in row echelon form has zeros below and above each leading 1

$$\begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Elimination Methods

Example 2:

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Remark:

The last matrix is in reduced row echelon form. The procedure (or algorithm) we have just described for reducing a matrix to reduced row echelon form is called **Gauss–Jordan elimination**

Example 3:

Solve by Gauss–Jordan elimination.

$$
\begin{aligned}
x_1 + 3x_2 - 2x_3 \phantom{{}- 2x_4} + 2x_5 \phantom{{}- 3x_6} &= 0 \\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\
5x_3 + 10x_4 \phantom{{}+ 4x_5} + 15x_6 &= 5 \\
2x_1 + 6x_2 \phantom{{}- 5x_3} + 8x_4 + 4x_5 + 18x_6 &= 6
\end{aligned}
$$

Homogeneous Linear Systems

A system of linear equations is said to be **homogeneous** if the constant terms are all zero; that is, the system has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

Every homogeneous system of linear equations is consistent because all such systems have $x_1 = 0, x_2 = 0, \dots \dots, x_n = 0$ as a solution .

Example 4:

Use Gauss–Jordan elimination to solve the homogeneous linear system

$$\begin{aligned}
x_1 + 3x_2 - 2x_3 \phantom{{}- 2x_4} + 2x_5 \phantom{{}- 3x_6} &= 0\\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0\\
5x_3 + 10x_4 \phantom{{}+ 4x_5} + 15x_6 &= 0\\
2x_1 + 6x_2 \phantom{{}- 5x_3} + 8x_4 + 4x_5 + 18x_6 &= 0
\end{aligned}$$