1.3 Matrices and Matrix Operations

Matrix

DEFINITION 1

- A ***matrix*** is a rectangular array of numbers.
- A matrix with m rows and n columns is called an ***m x n matrix.***
- ***The size of*** an mxn matrix is mxn
- The plural of matrix is matrices.
- ***A square matrix*** is a matrix with the same number of rows and columns

Example 1

The matrix $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix}$ size is………………….

A Column Matrix & Raw Matrix

- A matrix with only one column is called a ***column matrix.***

Example:

It is a 3x1 column matrix

$$\begin{bmatrix} 6 \\ 8 \\ 4 \end{bmatrix}$$

- A matrix with only one raw is called a

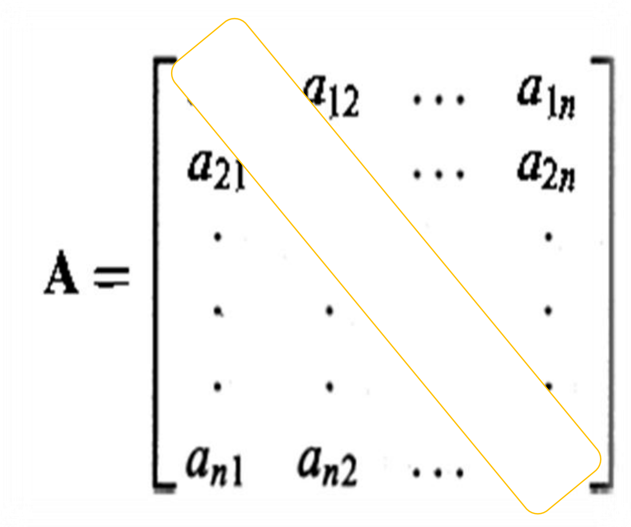
raw matrix.

Example:

$$\begin{bmatrix} -9 & 9 & 0 \end{bmatrix}$$

It is a 1x3 row matrix

Main Diagonal



The shaded entries $(a)_{11}$, $(a)_{22}$, …., $(a)_{nn}$ are said to be on the main diagonal of A.

Example:

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix}$$

$(a)_{11} =$, $(a)_{12} =$

$(a)_{21} =$, $(a)_{22} =$

$(a)_{31} =$, $(a)_{32} =$

Equality of Tow Matrices

Definition

Two matrices are defined to be ***equal*** if

- They have the ***same size*** and
- Their corresponding ***entries are equal***.

Example:

Consider the matrices

$$A = \begin{bmatrix} 2 & 1 \\ 3 & x \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \end{bmatrix}$$

- Is A=B , B = C , A = C.

Matrix Operations

- **Addition and Subtraction.**
- **Scalar Multiples.**
- **Multiplying Matrices.**

Addition and Subtraction of matrices

DEFINITION 3

- If A and B are matrices of the **same size**, then the ***sum A+B*** is the matrix obtained by ***adding*** the entries of B to the corresponding entries of A, and
- The ***difference A-B*** is the matrix obtained by ***subtracting*** the entries of B from the corresponding entries of A.
- **Matrices of different sizes cannot be added or subtracted.**

EXAMPLE 2

Consider the matrices

$$A = \begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 4 & -2 & 7 & 0 \end{bmatrix} \quad B = \begin{bmatrix} -4 & 3 & 5 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & 2 & -4 & 5 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

Find: A+B , A-B , A+C, B+C, A-C, and B-C

Scalar Multiples.

DEFINITION 3

If $\boldsymbol{A}$ is any matrix and $\boldsymbol{c}$ is any scalar, then the ***product cA*** is the matrix obtained by multiplying each entry of the matrix $\boldsymbol{A}$ by $\boldsymbol{c}$.

The matrix $\boldsymbol{cA}$ is said to be a ***scalar multiple*** of $\boldsymbol{A}$.

In matrix notation,

If $A = [a_{ij}]$, then $cA = c[a_{ij}], = [ca_{ij}]$.

Example:

For the matrices

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 3 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 2 & 7 \\ -1 & 3 & -5 \end{bmatrix} \quad C = \begin{bmatrix} 9 & -6 & 3 \\ 3 & 0 & 12 \end{bmatrix}$$

Find: 2A , - B , (1 ⁄3)C

Multiplying Matrices.

DEFINITION 4

Let A be an m x k matrix and B be a k x n matrix. ***The product of A and B***, denoted by ***AB***, is

the m x n matrix with its (i,j)th entry equal to the sum of the products of the corresponding

elements from the ith row of A and the jth column of B.

In other words, if AB = $[c_{ij}]$ then $c_{ij} = a_{1j}\, b_{1j} + a_{i2}b_{2i} + \ldots + a_{ik}b_{ki}$.

Determining Whether a product Is Defined

Suppose that A, B, and C are matrices with the following sizes:

A	B	C
3x4	4x7	7x3

Then

- *AB* is defined and is a 3x7 matrix;
- *BC* is defined and is a 4x3 matrix;
- *CA* is defined and is a 7x4 matrix.
- The products *AC, CB*, and *BA* are all undefined.

EXAMPLE 3

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix}.$$

Find **AB** if it is defined.

$$AB = \begin{bmatrix} (1\times 2)+(0\times 1)+(4\times 3) & (1\times 4)+(0\times 1)+(4\times 0) \\ (2\times 2)+(1\times 1)+(1\times 3) & (2\times 4)+(1\times 1)+(1\times 0) \\ (3\times 2)+(1\times 1)+(0\times 3) & (3\times 4)+(1\times 1)+(0\times 0) \\ (0\times 2)+(2\times 1)+(2\times 3) & (0\times 4)+(2\times 1)+(2\times 0) \end{bmatrix}_{4\times 2} = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ 8 & 2 \end{bmatrix}_{4\times 2}$$

Coution!

AB≠BA

EXAMPLE 4

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Does $\mathbf{AB} = \mathbf{BA}$?

The Identity Matrices

DEFINITION 5 The *identity matrix of order n* is the $n \times n$ matrix $\mathbf{I}_n = [\delta_{ij}]$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Hence

$$\mathbf{I}_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

Multiplying a matrix by an appropriately sized identity matrix does not change this matrix. In other words, when $\mathbf{A}$ is an $m \times n$ matrix, we have

$$\mathbf{A}\mathbf{I}_n = \mathbf{I}_m\mathbf{A} = \mathbf{A}.$$

The Zero Matrices

Definition

A matrix whose entries are all zero is called a ***zero matrix***.

We will denote a zero matrix by 0 unless it is important to specify its size, in which case we will denote the mxn zero matrix by 0_{mxn} .

Examples:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix};\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix};\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix};[0]$$

Remark

- If A and 0 are matrices ***of the same sizes***, then

$A+0= 0+A= A$ **&** ***$A-0 = 0-A = A$***

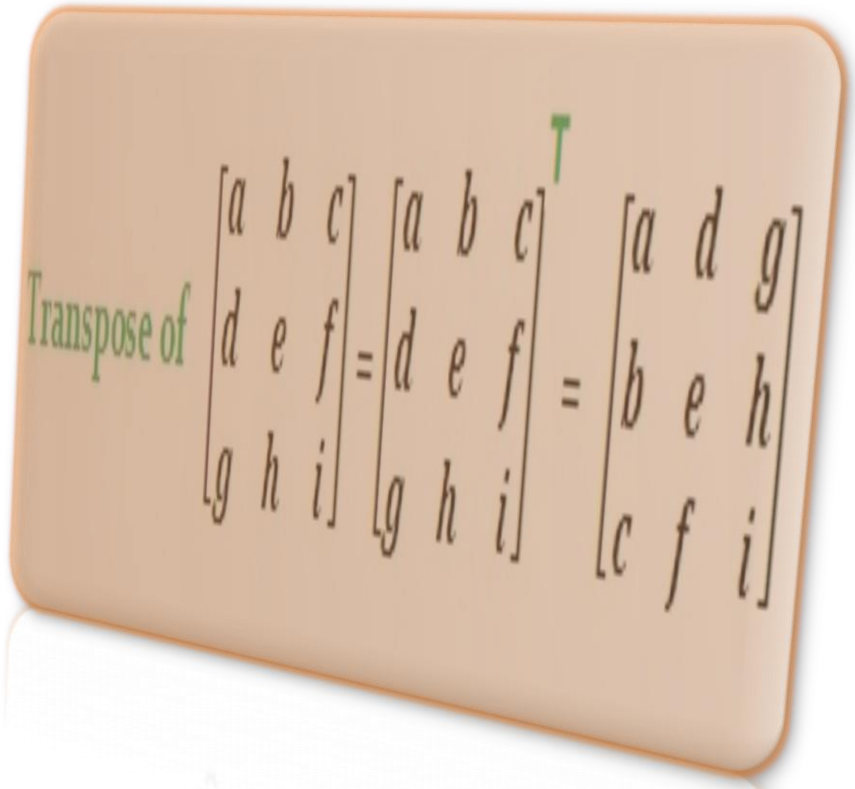
- If A and 0 are matrices of a ***different sizes***, then

$A+0$, $0+A$, $A-0$ & $0-A$ ***are not defined***.

Transpose of Matrix

DEFINITION 6

If A is any *mxn* matrix, then *the transpose* of *A*, denoted by A^T, is defined to be *the nxm* matrix that results *by interchanging rows and columns of matrix A*; that is; the first column of A^T is the first row of A, and the second column of A^T is the column row of A, and so forth.



Example:

The following are some examples of matrices and their transposes.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}; B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 3 & 5 \end{bmatrix}, \qquad D = \begin{bmatrix} 4 \end{bmatrix}$$

The Trace of a Matrix

Definition

If A is a **square matrix**, then ***the trace of A***, denoted by ***tr(A),*** is defined to be ***the sum of the entries on the main diagonal of A.*** The **trace of A is undefined if A is not a square** matrix.

Example:

Find trace of the following matrix $A=\begin{bmatrix} 2 & -5 & 5 \\ 0 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

Solution:

tr(A)= 2+3+1=6.

Exercise Set 1.3

▶ In Exercises 1–2, suppose that A, B, C, D, and E are matrices with the following sizes:

A	B	C	D	E
(4×5)	(4×5)	(5×2)	(4×2)	(5×4)

In each part, determine whether the given matrix expression is defined. For those that are defined, give the size of the resulting matrix. ◀

1. (a) BA (b) AB^T (c) $AC+D$
(d) $E(AC)$ (e) $A-3E^T$ (f) $E(5B+A)$

2. (a) CD^T (b) DC (c) $BC-3D$
(d) $D^T(BE)$ (e) B^TD+ED (f) BA^T+D

▶ In Exercises 3–6, use the following matrices to compute the indicated expression if it is defined.

$$A=\begin{bmatrix}3 & 0\\ -1 & 2\\ 1 & 1\end{bmatrix},\quad B=\begin{bmatrix}4 & -1\\ 0 & 2\end{bmatrix},\quad C=\begin{bmatrix}1 & 4 & 2\\ 3 & 1 & 5\end{bmatrix},$$

$$D=\begin{bmatrix}1 & 5 & 2\\ -1 & 0 & 1\\ 3 & 2 & 4\end{bmatrix},\quad E=\begin{bmatrix}6 & 1 & 3\\ -1 & 1 & 2\\ 4 & 1 & 3\end{bmatrix}$$ ◀

3. (a) $D+E$ (b) $D-E$ (c) $5A$
(d) $-7C$ (e) $2B-C$ (f) $4E-2D$
(g) $-3(D+2E)$ (h) $A-A$ (i) $\text{tr}(D)$
(j) $\text{tr}(D-3E)$ (k) $4\,\text{tr}(7B)$ (l) $\text{tr}(A)$

4. (a) $2A^T+C$ (b) D^T-E^T (c) $(D-E)^T$
(d) B^T+5C^T (e) $\frac{1}{2}C^T-\frac{1}{4}A$ (f) $B-B^T$
(g) $2E^T-3D^T$ (h) $(2E^T-3D^T)^T$ (i) $(CD)E$
(j) $C(BA)$ (k) $\text{tr}(DE^T)$ (l) $\text{tr}(BC)$

5. (a) AB (b) BA (c) $(3E)D$
(d) $(AB)C$ (e) $A(BC)$ (f) CC^T
(g) $(DA)^T$ (h) $(C^TB)A^T$ (i) $\text{tr}(DD^T)$
(j) $\text{tr}(4E^T-D)$ (k) $\text{tr}(C^TA^T+2E^T)$ (l) $\text{tr}((EC^T)^TA)$

6. (a) $(2D^T-E)A$ (b) $(4B)C+2B$
(c) $(-AC)^T+5D^T$ (d) $(BA^T-2C)^T$
(e) $B^T(CC^T-A^TA)$ (f) $D^TE^T-(ED)^T$

▶ In Exercises 7–8, use the following matrices and either the row method or the column method, as appropriate, to find the indicated row or column.

$$A=\begin{bmatrix}3 & -2 & 7\\ 6 & 5 & 4\\ 0 & 4 & 9\end{bmatrix}\quad\text{and}\quad B=\begin{bmatrix}6 & -2 & 4\\ 0 & 1 & 3\\ 7 & 7 & 5\end{bmatrix}$$ ◀

7. (a) the first row of AB (b) the third row of AB
(c) the second column of AB (d) the first column of BA
(e) the third row of AA (f) the third column of AA

8. (a) the first column of AB (b) the third column of BB

(c) the second row of BB (d) the first column of AA

(e) the third column of AB (f) the first row of BA

In Exercises 9–10, use matrices A and B from Exercises 7–8.

9. (a) Express each column vector of AA as a linear combination of the column vectors of A.

(b) Express each column vector of BB as a linear combination of the column vectors of B.

10. (a) Express each column vector of AB as a linear combination of the column vectors of A.

(b) Express each column vector of BA as a linear combination of the column vectors of B.

In each part of Exercises 11–12, find matrices A, $\mathbf{x}$, and $\mathbf{b}$ that express the given linear system as a single matrix equation $A\mathbf{x} = \mathbf{b}$, and write out this matrix equation.

11. (a)
$$\begin{aligned} 2x_1 - 3x_2 + 5x_3 &= 7 \\ 9x_1 - x_2 + x_3 &= -1 \\ x_1 + 5x_2 + 4x_3 &= 0 \end{aligned}$$

(b)
$$\begin{aligned} 4x_1 \qquad - 3x_3 + x_4 &= 1 \\ 5x_1 + x_2 \qquad\qquad - 8x_4 &= 3 \\ 2x_1 - 5x_2 + 9x_3 - x_4 &= 0 \\ 3x_2 - x_3 + 7x_4 &= 2 \end{aligned}$$

12. (a)
$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= -3 \\ 2x_1 + x_2 \qquad &= 0 \\ - 3x_2 + 4x_3 &= 1 \\ x_1 \qquad + x_3 &= 5 \end{aligned}$$

(b)
$$\begin{aligned} 3x_1 + 3x_2 + 3x_3 &= -3 \\ -x_1 - 5x_2 - 2x_3 &= 3 \\ - 4x_2 + x_3 &= 0 \end{aligned}$$

In each part of Exercises 13–14, express the matrix equation as a system of linear equations.

13. (a) $\begin{bmatrix} 5 & 6 & -7 \\ -1 & -2 & 3 \\ 0 & 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 5 & -3 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -9 \end{bmatrix}$

14. (a) $\begin{bmatrix} 3 & -1 & 2 \\ 4 & 3 & 7 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & -2 & 0 & 1 \\ 5 & 0 & 2 & -2 \\ 3 & 1 & 4 & 7 \\ -2 & 5 & 1 & 6 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

In Exercises 15–16, find all values of k, if any, that satisfy the equation.

15. $\begin{bmatrix} k & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} k \\ 1 \\ 1 \end{bmatrix} = 0$

16. $\begin{bmatrix} 2 & 2 & k \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 3 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ k \end{bmatrix} = 0$

In Exercises 17–20, use the column-row expansion of AB to express this product as a sum of matrices.

17. $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 2 \\ -2 & 3 & 1 \end{bmatrix}$

18. $A = \begin{bmatrix} 0 & -2 \\ 4 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 4 & 1 \\ -3 & 0 & 2 \end{bmatrix}$

19. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

20. $A = \begin{bmatrix} 0 & 4 & 2 \\ 1 & -2 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ 4 & 0 \\ 1 & -1 \end{bmatrix}$

21. For the linear system in Example 5 of Section 1.2, express the general solution that we obtained in that example as a linear combination of column vectors that contain only numerical entries. [*Suggestion:* Rewrite the general solution as a single column vector, then write that column vector as a sum of column vectors each of which contains at most one parameter, and then factor out the parameters.]

22. Follow the directions of Exercise 21 for the linear system in Example 6 of Section 1.2.

In Exercises 23–24, solve the matrix equation for a, b, c, and d.

23. $\begin{bmatrix} a & 3 \\ -1 & a+b \end{bmatrix} = \begin{bmatrix} 4 & d-2c \\ d+2c & -2 \end{bmatrix}$

24. $\begin{bmatrix} a-b & b+a \\ 3d+c & 2d-c \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ 7 & 6 \end{bmatrix}$

25. (a) Show that if A has a row of zeros and B is any matrix for which AB is defined, then AB also has a row of zeros.

(b) Find a similar result involving a column of zeros.

26. In each part, find a 6×6 matrix $[a_{ij}]$ that satisfies the stated condition. Make your answers as general as possible by using letters rather than specific numbers for the nonzero entries.

(a) $a_{ij} = 0$ if $i \neq j$ (b) $a_{ij} = 0$ if $i > j$

(c) $a_{ij} = 0$ if $i < j$ (d) $a_{ij} = 0$ if $|i - j| > 1$