UNIT 1A

Systems of Linear Equations and Matrices

Unit 1A · Sections 1.1 – 1.2

MATH 244 — Linear Algebra
Elementary Linear Algebra, Anton & Rorres, 11th edition
King Saud University · Department of Mathematics

LEARNING OUTCOMES

What you should be able to do

Section 1.1

- Decide whether an equation is linear in the given variables.
- Write the augmented matrix of a linear system and recover the system from its augmented matrix.
- Describe the solution set of a system geometrically and decide whether it is consistent.
- Apply the three elementary row operations.

Section 1.2

- Recognise row echelon form and reduced row echelon form.
- Solve a linear system by Gauss–Jordan elimination and by Gaussian elimination with back-substitution.
- Identify leading and free variables and write a general solution in parametric form.
- Use the Free Variable Theorem to decide when a homogeneous system has nontrivial solutions.

SECTION 1.1

Introduction to Systems of Linear Equations

GOAL — Introduce the language of linear systems and a first, informal method for solving them.

1. Linear equations and linear systems
2. Solutions and n-tuples
3. Geometry: zero, one, or infinitely many solutions
4. Augmented matrices
5. Elementary row operations

DEFINITION

Linear equation

A linear equation in the n variables $x_1,\ x_2,\ \dots,\ x_n$ is one that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \qquad (1)$$

where $a_1,\ a_2,\ \dots,\ a_n$ and b are constants and the a's are not all zero.

Familiar special cases

$$a_1x + a_2y = b \qquad (a_1,\ a_2 \text{ not both } 0)$$

$$a_1x + a_2y + a_3z = b \qquad (a_1,\ a_2,\ a_3 \text{ not all } 0)$$

- If $b = 0$ the equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$ is called homogeneous.

EXAMPLE 1

Linear or not linear?

A linear equation contains no products or roots of variables; every variable appears to the first power only and never inside a trigonometric, logarithmic, or exponential function.

LINEAR

$$x + 3y = 7$$

$$1/2\, x - y + 3z = -1$$

$$x_1 - 2x_2 - 3x_3 + x_4 = 0$$

$$x_1 + x_2 + \cdots + x_n = 1$$

NOT LINEAR

$$x + 3y^2 = 4$$

$$\sin x + y = 0$$

$$3x + 2y - xy = 5$$

$$\sqrt{x_1} + 2x_2 + x_3 = 1$$

Solutions, n-tuples and the solution set

A **solution** of a linear equation in n unknowns

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

is a list of n numbers $x_1 = s_1,\ x_2 = s_2,\ \dots,\ x_n = s_n$ that makes the equation a true statement:

$$a_1s_1 + a_2s_2 + \cdots + a_ns_n = b$$

- The list $(s_1, s_2, \dots, s_n)$ is an n**-tuple** — an ordered pair when $n = 2$, an ordered triple when $n = 3$.

- The **solution set** is the set of ALL solutions of the equation.

EXAMPLE

Parametric representation of a solution set

The equation $x_1 + 2x_2 = 4$ has $(2,1)$ as one solution, and infinitely many others. Solving for x_1 gives $x_1 = 4 - 2x_2$, so x_2 may be chosen freely: it is a **free variable**.

Setting $x_2 = t$ — the **parameter** — gives the general solution

$$x_1 = 4 - 2t, \qquad x_2 = t, \qquad t \in \mathbb{R}$$

and the solution set written as a set is

$$\{(4 - 2t,\ t) : \ t \in \mathbb{R}\}$$

The choice of free variable is not forced. Taking $x_1 = s$ instead gives $x_2 = 2 - \frac{s}{2}$ and the set $\{\left(s,\ 2 - \frac{s}{2}\right) : s \in \mathbb{R}\}$ — the same line, differently labelled.

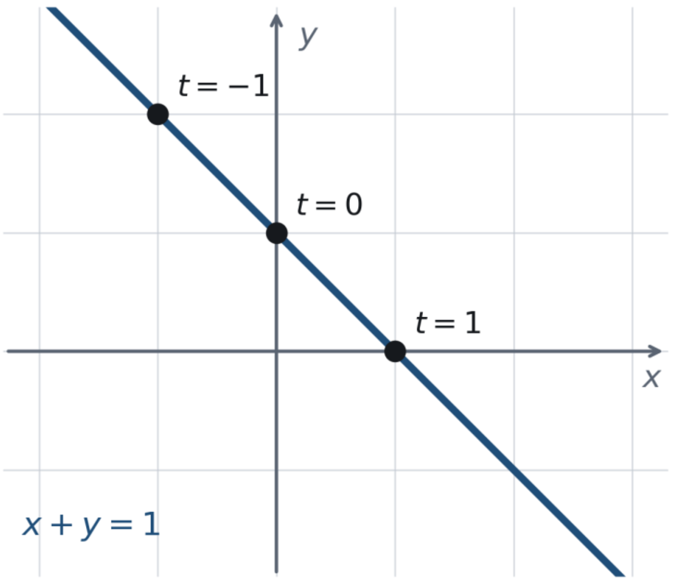
One equation, two unknowns: a line

The equation $x + y = 1$ describes a **line** in the plane; written as $y = 1 - x$ it is the **implicit equation** of that line.

The same line has a **parametric form** in $\mathbb{R}^2$:

$$(x, y) = (t,\ 1 - t), \qquad t \in \mathbb{R}$$

- Every point of the line is $(t,\ 1 - t)$ for exactly one real t.
- One parameter is enough: a line is one-dimensional, even though it lives in $\mathbb{R}^2$.



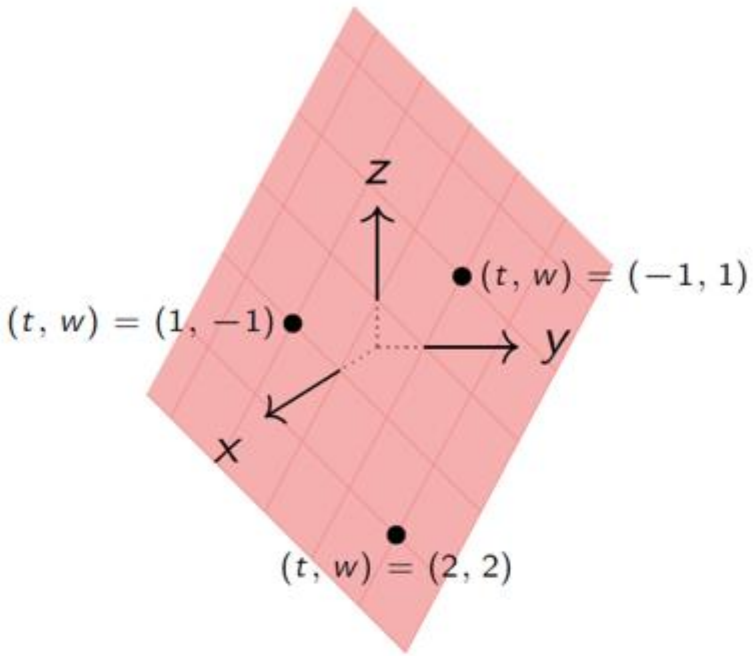
One equation, three unknowns: a plane

The equation $x + y + z = 1$ describes a **plane** in space — this is its **implicit equation**.

In parametric form:

$$(x, y, z) = (t,\ w,\ 1 - t - w), \qquad t, w \in \mathbb{R}$$

- Two parameters are needed: a plane is two-dimensional.
- So $\mathbb{R}^2$ labels the points of a plane sitting inside $\mathbb{R}^3$.



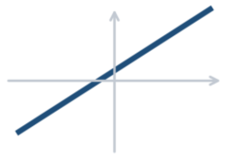
One equation in one unknown: $ax = b$

	$a \neq 0$	$a = 0$	
		$b = 0$	$b \neq 0$
Equation	$ax = b$	$0 \cdot x = 0$	$0 \cdot x = b \neq 0$
General Solution	$x = \dfrac{b}{a}$	x is any real number	no solution
Solution set	$\left\{\dfrac{b}{a}\right\}$	$\mathbb{R}$	$\emptyset$

> Note: the possibilities for solutions are: exactly one solution, infinitely many, or none.

CASE ANALYSIS

One equation in two unknowns: $a_1x + a_2y = b$

	At least one of a_1 or $a_2 \neq 0$	$a_1 = a_2 = 0$	
		$b = 0$	$b \neq 0$
Equation	$a_1x + a_2y = b$ the equation of a line	$0 \cdot x + 0 \cdot y = 0$	$0 \cdot x + 0 \cdot y = b \neq 0$
General Solution	use a parametric representation	any pair of real numbers	no solution
Solution set	all points on the line 	$\mathbb{R}^2$ 	$\emptyset$

A single equation in two unknowns never has exactly one solution: with a nonzero coefficient there is always a whole line of them.

One equation in n unknowns: $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ with $n \geq 2$.

	at least one $a_i \neq 0$	$a_1 = a_2 = \cdots = a_n = 0$	
		$b = 0$	$b \neq 0$
Equation	$a_1x_1 + \cdots + a_nx_n = b$	$0 \cdot x_1 + \cdots + 0 \cdot x_n = 0$	$0 \cdot x_1 + \cdots + 0 \cdot x_n = b \neq 0$
General Solution	solve for one x_i with $a_i \neq 0$; the other $n-1$ unknowns are free: $x_i = \frac{b}{a_i} - \frac{a_1}{a_i}x_1 - \cdots - \frac{a_{i-1}}{a_i}x_{i-1} - \frac{a_{i+1}}{a_i}x_{i+1} - \cdots - \frac{a_n}{a_i}x_n$	any n-tuple of real numbers	no solution
Solution set	all points of a hyperplane — $n-1$ parameters	$\mathbb{R}^n$	$\emptyset$

DEFINITION

Linear system and its general form

A finite set of linear equations is a system of linear equations, or a linear system. The variables are called the unknowns.

A general system of m equations in n unknowns

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \qquad (7)$$

- A **solution** of a system of linear equations is a sequence of numbers $s_1, s_2, \ldots, s_n$ that satisfies all equations in the system.
- **The solution set** of a system of linear equations is the collection of all solutions.
- **Solving** a system of linear equations means finding the solution set of the system.
- A system is call **consistent** if it has at least one solution and it is called **inconsistent** if it has no solution.

ORGANISING IDEA

The two questions we ask of every system

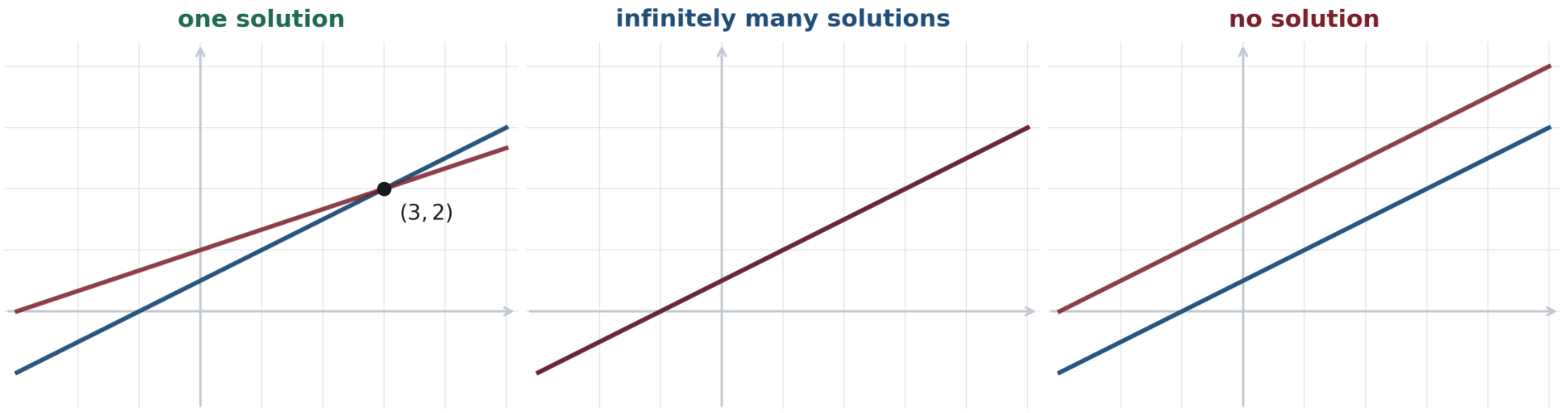
Everything in Chapter 1 is aimed at two questions. Keep them in view — Section 1.2 will answer both by inspection of a row echelon form.

> EXISTENCE. Is the system consistent — does at least one solution exist?
> UNIQUENESS. If a solution exists, is it the only one?

Two equations, two unknowns: three pictures

Each equation is a line, so the solution set of the system is the intersection of two lines. Only three things can happen.

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned} \qquad \begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 1 \end{aligned} \qquad \begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \end{aligned}$$

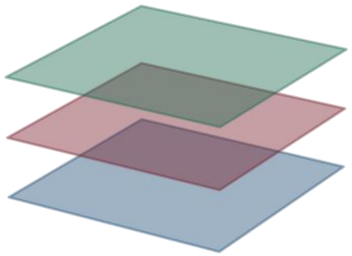


Non-parallel lines meet once; identical lines meet everywhere; parallel distinct lines never meet.

How many solutions can a linear system have?

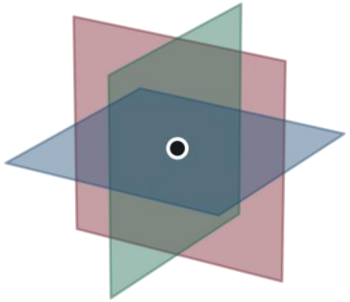
Three equations in three unknowns are three planes; the solutions are the points common to all three. Again only three outcomes occur.

No solution



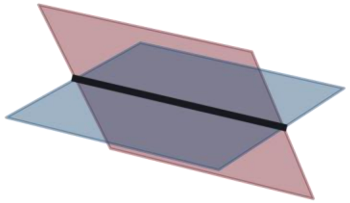
three parallel planes

One solution



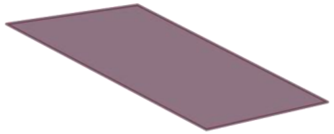
they meet in a point

Infinitely many



they meet in a line

Infinitely many



the planes coincide

> Every system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.

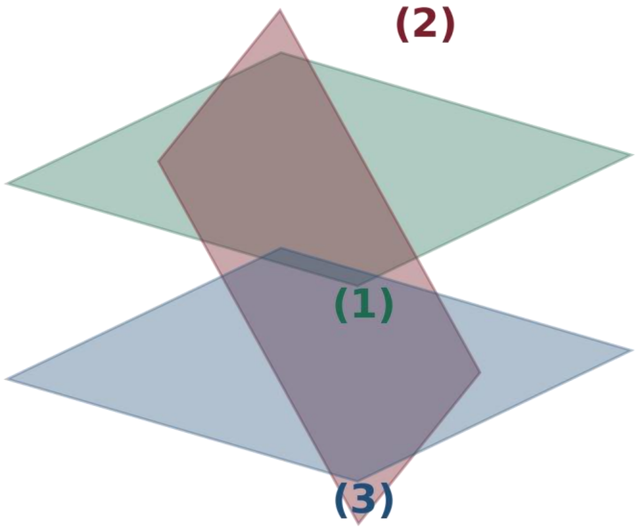
- So a consistent system can never have exactly two, or exactly seventeen, solutions.

EXAMPLE

Three equations, three unknowns: reading the picture

Each equation is a plane in $\mathbb{R}^3$:

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 4 \\ 8x_1 - x_2 + 2x_3 &= 0 \\ -x_1 - 2x_2 + 3x_3 &= 1 \end{aligned}$$



- Equations (1) and (3) have opposite sign coefficients: adding them gives the impossible equation $0 = 5$.
- They are parallel and distinct, so no point lies on both — and the third plane cannot repair that. The system is **inconsistent**.

EXAMPLE

Back-substitution: solving a triangular system

A **triangular** system — one in row echelon form — is solved from the bottom up, one unknown at a time. That is **back-substitution**.

Two unknowns

$$\begin{aligned} x - 2y &= 5 \\ y &= -2 \end{aligned}$$

The second equation gives $y = -2$; then $x - 2(-2) = 5$, so $x = 1$. One solution: $(1, -2)$.

Three unknowns

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned}$$

From the bottom: $z = 2$; then $y + 3(2) = 5$ gives $y = -1$; then $x - 2(-1) + 3(2) = 9$ gives $x = 1$. Solution: $(1, -1, 2)$.

DEFINITION

Equivalent systems

Two linear systems are **EQUIVALENT** when they have exactly the same solution set.

Each of the following operations produces equivalent systems.

- Interchange two equations, say equation.
- Multiply an equation through by a nonzero constant c.
- Add a constant times one equation to another.

Each of these three operations is reversible:

- interchanging two rows is undone by interchanging them back,
- scaling by c is undone by scaling by $1/c$, and
- adding c times row i to row j is undone by adding $-c$ times row i to row j.

- Reversibility means anything satisfying the new system also satisfies the old one, and conversely.
- A system can be solved by a chain of EQUIVALENT systems until one is simple enough to read

EXAMPLE 3

A system with no solution

Solve

$$\begin{aligned} x + y &= 4 \\ 3x + 3y &= 6 \end{aligned}$$

Solution

Add -3 times the first equation to the second:

$$\begin{aligned} x + y &= 4 \\ 0 &= -6 \end{aligned}$$

- The second equation is contradictory, so the system has no solution — it is inconsistent.
- Geometrically the two lines are parallel and distinct: same slope, different y-intercepts.

EXAMPLE 5

Two parameters: three coincident planes

Solve

$$\begin{aligned} x - y + 2z &= 5 \\ 2x - 2y + 4z &= 10 \\ 3x - 3y + 6z &= 15 \end{aligned}$$

Solution

Equations two and three are multiples of the first, so the three planes coincide and it suffices to solve

$$x - y + 2z = 5 \qquad (9)$$

Solve for x and assign **parameters** $y = r$, $z = s$ we have a **general solution**:

$$x = 5 + r - 2s, \qquad y = r, \qquad z = s$$

- Taking $r = 1$, $s = 0$ gives the **particular solution** $(6,1,0)$.

EXAMPLE

Elimination on the equations: a unique solution

Solve

$$
\begin{aligned}
x - 2y + 3z &= 9 \qquad (1)\\
-x + 3y &= -4 \qquad (2)\\
2x - 5y + 5z &= 17 \qquad (3)
\end{aligned}
$$

Eliminate x from equations (2) and (3)

Apply $A_{1,2}^{(1)}$ — add 1 ×(1) to (2) — and then $A_{1,3}^{(-2)}$:

$$
\begin{aligned}
x - 2y + 3z &= 9 \qquad (1)\\
y + 3z &= 5 \qquad (4)\\
-y - z &= -1 \qquad (5)
\end{aligned}
$$

> Only the equation named second by the symbol changes. In Section 1.2 the same symbols act on the ROWS of a matrix.

EXAMPLE

Elimination on the equations: finishing the job

Add 1 ×(4) to (5) — the operation $A_{4,5}^{(1)}$ — then halve the last equation with $M_3^{(1/2)}$:

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2z &= 4 \end{aligned} \quad \xrightarrow{M_3^{(1/2)}} \quad \begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned}$$

The system is now triangular, so back-substitute: $z = 2$, then $y = -1$, then $x = 1$.

$$x = 1, \qquad y = -1, \qquad z = 2$$

> Exactly one solution — the three planes meet in a single point. In Section 1.2 this same system is solved again in matrix form by Gauss-Jordan elimination: same operations, far less writing.

EXAMPLE

An inconsistent system: the signal 0 = k

Solve

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1 && (1) \\ 2x_1 - x_2 - 2x_3 &= 2 && (2) \\ x_1 + 2x_2 - 3x_3 &= -1 && (3) \end{aligned}$$

Apply $A_{1,2}^{(-2)}$ and $A_{1,3}^{(-1)}$ to clear x_1 from the last two equations:

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1 \\ 5x_2 - 4x_3 &= 0 && (4) \\ 5x_2 - 4x_3 &= -2 && (5) \end{aligned} \quad \xrightarrow{A_{4,5}^{(-1)}} \quad \begin{aligned} x_1 - 3x_2 + x_3 &= 1 \\ 5x_2 - 4x_3 &= 0 \\ 0 &= -2 \end{aligned}$$

> An equation of the form $0 = k$ with $k \neq 0$ cannot hold. The system has no solution: it is **inconsistent**, and its solution set is $\emptyset$.

EXAMPLE

Infinitely many solutions: a redundant equation

Solve

$$\begin{aligned} x_2 - x_3 &= 0 && (1) \\ x_1 - 3x_3 &= -1 && (2) \\ -x_1 + 3x_2 &= 1 && (3) \end{aligned}$$

Interchange the first two equations with $I_{1,2}$ so that x_1 leads, then apply $A_{1,3}^{(1)}$:

$$\begin{aligned} x_1 - 3x_3 &= -1 \\ x_2 - x_3 &= 0 \\ -x_1 + 3x_2 &= 1 \end{aligned} \quad \xrightarrow{A_{1,3}^{(1)}} \quad \begin{aligned} x_1 - 3x_3 &= -1 \\ x_2 - x_3 &= 0 \\ 3x_2 - 3x_3 &= 0 \quad (4) \end{aligned}$$

Equation (4) is three times equation (2): it says nothing new and can be discarded.

EXAMPLE

Infinitely many solutions: writing the answer

Two equations are left for three unknowns:

$$\begin{aligned} x_1 - 3x_3 &= -1 \\ x_2 - x_3 &= 0 \end{aligned} \qquad \Rightarrow \qquad x_1 = -1 + 3x_3, \quad x_2 = x_3$$

Nothing pins down x_3, so it is a free variable. Set $x_3 = t$:

$$x_1 = 3t - 1, \qquad x_2 = t, \qquad x_3 = t, \qquad t \in \mathbb{R}$$

so the solution set is

$$\{\,(3t - 1,\ t,\ t)\ :\ t \in \mathbb{R}\,\}$$

> One free variable gives a line of solutions inside $\mathbb{R}^3$ — the three planes share a common line, exactly the second picture in the exercise above.

DEFINITION

The augmented matrix

A system can be abbreviated by the rectangular array of its numbers:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right] = [\mathrm{A}|\mathrm{b}]$$

The system

$$\begin{array}{rcl} x_1 + x_2 + 2x_3 & = & 9 \\ 2x_1 + 4x_2 - 3x_3 & = & 1 \\ 3x_1 + 6x_2 - 5x_3 & = & 0 \end{array}$$

Its augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right]$$

- Write the unknowns in the same order in every equation, and put a 0 wherever an unknown is missing.

EXAMPLE 6

Row operations, side by side (part 1 of 2)

Operating on the equations

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

Add $-2 \times$ row 1 to row 2, then $-3 \times$ row 1 to row 3:

$$\begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3y - 11z &= -27 \end{aligned}$$

Multiply equation 2 by $\frac{1}{2}$:

$$\begin{aligned} x + y + 2z &= 9 \\ y - \frac{7}{2}z &= -\frac{17}{2} \\ 3y - 11z &= -27 \end{aligned}$$

Operating on the rows

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right]$$

Same two operations on the rows:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array}\right]$$

Multiply row 2 by $\frac{1}{2}$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{array}\right]$$

EXAMPLE 6

Row operations, side by side (part 2 of 2)

Add $-3 \times$ row 2 to row 3, then multiply row 3 by -2:

$$\begin{aligned} x + y + 2z &= 9 \\ y - \frac{7}{2}z &= -\frac{17}{2} \\ z &= 3 \end{aligned}$$

The same steps on the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Clear above the leading 1's:

$$\begin{aligned} x &= 1 \\ y &= 2 \\ z &= 3 \end{aligned}$$

and finally

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

- The solution $x = 1$, $y = 2$, $z = 3$ is now evident; as a triple, $(1,2,3)$.

DEFINITION

Elementary row operations

Recall the basic method of solution is to perform operations, producing simpler and simpler equivalent systems:

- Multiply an equation through by a nonzero constant.
- Interchange two equations.
- Add a constant times one equation to another.

Rows of the augmented matrix correspond to equations, so the same three operations on rows are the **elementary row operations**:

> - Multiply a row through by a nonzero constant $c \neq 0$.
> - Interchange two rows.
> - Add a constant times one row to another.

- Note the restriction $c \neq 0$: multiplying a row by zero is NOT an elementary row operation, since it destroys information.

NOTATION

Recording a row operation

Each operation gets a symbol, so a whole reduction can be read back and checked line by line.

Interchange

$$I_{i,j} \equiv R_i \leftrightarrow R_j$$

Scale by $k \neq 0$

$$M_i^{(k)} \equiv (k)R_i \to R_i$$

Add a multiple

$$A_{i,j}^{(k)} \equiv (k)R_i + R_j \to R_j$$

Read $A_{i,j}^{(k)}$ as: add k times row i to row j; only ROW j changes.

SECTION 1.2

Gaussian Elimination

GOAL — Turn the informal row work of §1.1 into a systematic algorithm that solves any linear system.

1. Row echelon and reduced row echelon form
2. Leading variables, free variables, general solution
3. Gauss–Jordan elimination: the six steps
4. Gaussian elimination with back-substitution
5. Homogeneous systems and the Free Variable Theorem

DEFINITION

Row echelon form and reduced row echelon form

A matrix is in **REDUCED ROW ECHELON FORM** when it has all four properties:

- In every nonzero row, the first nonzero entry is a 1 — a leading 1.
- All rows of zeros are grouped together at the bottom.
- In two successive nonzero rows, the leading 1 of the lower row is farther to the right.
- Each column containing a leading 1 has zeros everywhere else.

> A matrix with the **first three properties** only is in **ROW ECHELON FORM**.

$$\begin{bmatrix} \boxed{1} & * & 0 & * & 0 & * \\ 0 & 0 & \boxed{1} & * & 0 & * \\ 0 & 0 & 0 & 0 & \boxed{1} & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced row echelon form: leading 1's step down and to the right,
with zeros below the staircase and zeros above every leading 1.

NOTE. Every reduced row echelon matrix is in row echelon form, but not conversely.

EXAMPLE 1

Which form is it?

In reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In row echelon form, but NOT reduced

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- In the last two matrices the entries above some leading 1 are not all zero, so property 4 fails.

VOCABULARY

Pivot positions and pivot columns

The leading entries of a row echelon form sit in the same places no matter which sequence of row operations produced it. Those places have a name.

> A **PIVOT POSITION** of a matrix A is a location that holds a leading 1 in the reduced row echelon form of A. A column containing a pivot position is a **PIVOT COLUMN**.

In

$$\begin{bmatrix} 0 & \boxed{\begin{matrix}1\\0\\0\end{matrix}} & -2 & \boxed{\begin{matrix}0\\1\\0\end{matrix}} & 1 \\ 0 & & 0 & & 3 \\ 0 & & 0 & & 0 \end{bmatrix}$$

the pivots are in columns 2 and 4.

Columns 1, 3 and 5 are non-pivot columns.

- Pivot columns of the coefficient part correspond to leading variables; non-pivot columns correspond to free variables. This single picture will keep returning throughout the course.

THEOREM

The reduced form is unique — the unreduced one is not

> Every matrix is row equivalent to ONE AND ONLY ONE matrix in reduced row echelon form.

- So $\mathbf{rref(A)}$ is a genuine property of A: two students who choose completely different sequences of row operations must finish with the same matrix.
- Row echelon form $\mathbf{ref(A)}$ is NOT unique. Adding one row to a higher row, usually produces a different row echelon matrix for the same A.
- Because the leading 1's of the reduced form are pinned down, the PIVOT POSITIONS are the same in every row echelon form of A — this is what makes the pivot language legitimate.

EXAMPLE 3

Reading off a unique solution

The augmented matrix of a system in x_1, x_2, x_3, x_4 has been reduced to

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right]$$

which is in reduced row echelon form and corresponds to

$$x_1 = 3, \quad x_2 = -1, \quad x_3 = 0, \quad x_4 = 5$$

- The solution is unique and may be written as the 4-tuple $(3, -1, 0, 5)$.

EXAMPLE 4

Three systems in three unknowns (a) and (b)

(a) Inconsistent

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The last row says

$$0x + 0y + 0z = 1$$

which no values satisfy, so the system is inconsistent.

(b) One free variable

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The zero row is discarded, leaving

$$\begin{aligned} x + 3z &= -1 \\ y - 4z &= 2 \end{aligned}$$

Leading variables x, y; free variable $z = t$:

$$x = -1 - 3t,\ \ y = 2 + 4t,\ \ z = t$$

EXAMPLE 4

Three systems in three unknowns (c)

Part (c) reduces to

$$\left[\begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Dropping the zero rows leaves the single equation

$$x - 5y + z = 4 \qquad (1)$$

whose solution set is a plane in 3-space. Solving for the leading variable x in terms of the free variables $y = s$ and $z = t$ gives the parametric form

$$x = 4 + 5s - t, \qquad y = s, \qquad z = t \qquad (2)$$

> DEFINITION 1. If a linear system has infinitely many solutions, a set of parametric equations from which all solutions are obtained by assigning values to the parameters is called a **GENERAL SOLUTION** of the system.

Leading variables and free variables

- The unknowns corresponding to columns with a leading 1 are the LEADING VARIABLES.
- All remaining unknowns are the FREE VARIABLES.
- Solve each equation for its leading variable in terms of the free variables, then assign the free variables arbitrary parameters $r, s, t, \ldots$

> Number of free variables = (number of unknowns) − (number of leading 1's).

- A row of the form $[0 \quad 0 \quad \cdots \quad 0 \mid 1]$ in the augmented matrix means the system is INCONSISTENT — check for it before naming any free variables.
- A row of all zeros imposes no condition and is simply discarded.

THEOREM

Existence and uniqueness, answered

The two questions from Section 1.1 now have a mechanical answer. Reduce the augmented matrix to any row echelon form and look at it.

EXISTENCE. The system is consistent if and only if the LAST column is not a pivot column — equivalently, if and only if no row has the form $[0 \quad \cdots \quad 0 \quad b]$ with $b \neq 0$.

UNIQUENESS. If it is consistent, the solution is unique when there are NO free variables, and there are infinitely many solutions when there is at least one free variable.

- This is also the promised proof that the count of solutions is 0, 1, or ∞: the three cases above are the only ones a row echelon form allows.
- Read existence first. If the system is inconsistent there is nothing to parametrize, so do not start naming free variables.

Gauss–Jordan elimination: the six steps

- STEP 1. Locate the leftmost column that is not entirely zeros.
- STEP 2. Interchange the top row with another row, if necessary, to bring a nonzero entry to the top of that column.
- STEP 3. If that entry is a, multiply the top row by $1/a$ to create a leading 1.
- STEP 4. Add suitable multiples of the top row to the rows below so that every entry below the leading 1 becomes zero.
- STEP 5. Cover the top row and repeat Steps 1–4 on the submatrix that remains, until the whole matrix is in row echelon form.

- STEP 6. Starting from the last nonzero row and working upward, add suitable multiples of each row to the rows above to create zeros ABOVE each leading 1.

> Steps 1–5 are the FORWARD PHASE and produce a row echelon form — this alone is GAUSSIAN ELIMINATION. Step 6 is the BACKWARD PHASE; all six steps together are GAUSS–JORDAN ELIMINATION.

Forward phase, step by step

Locate the leftmost nonzero column, bring a nonzero entry to the top, make it a leading 1, and clear below it. Then cover that row and repeat.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix} \overset{I_{1,2}}{\to} \begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

$$\overset{M_1^{(1/2)}}{\to} \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix} \overset{A_{1,3}^{(-2)}}{\to} \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

- Column 1 was the first nonzero column; after $M_1^{(1/2)}$ it carries a leading 1, and $A_{1,3}^{(-2)}$ clears the entry below it. Now cover row 1 and start again on the submatrix.

Finishing the forward phase, then the backward phase

$$\xrightarrow{M_2^{(-1/2)}} \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix} \xrightarrow{A_{2,3}^{(-5)}} \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

$$\xrightarrow{M_3^{(2)}} \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{A_{3,1}^{(-6)} \ A_{3,2}^{(7/2)}} \begin{bmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{A_{2,1}^{(5)}} \begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

- After $M_3^{(2)}$ the matrix is in ROW ECHELON FORM — the forward phase is done. The last two arrows are the backward phase and produce the REDUCED ROW ECHELON FORM.

EXAMPLE

Gauss–Jordan elimination: a unique solution

Solve

$$\begin{array}{rcrcrcr} x & - & 2y & + & 3z & = & 9 \\ -x & + & 3y & & & = & -4 \\ 2x & - & 5y & + & 5z & = & 17 \end{array}$$

From the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right] \overset{A_{1,2}^{(1)}\ A_{1,3}^{(-2)}}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{array}\right] \overset{A_{2,3}^{(1)}}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

$$\overset{M_3^{(1/2)}}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right] \overset{A_{3,2}^{(-3)}\ A_{3,1}^{(-3)}}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \overset{A_{2,1}^{(2)}}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

- The middle matrix is a row echelon form; the last is the reduced row echelon form, giving $x = 1$, $y = -1$, $z = 2$.

EXAMPLE

Gauss–Jordan elimination: infinitely many solutions

Solve

$$\begin{array}{ccccccc} 2x_1 & + & 4x_2 & - & 2x_3 & = & 0 \\ 3x_1 & + & 5x_2 & & & = & 1 \end{array}$$

From the augmented matrix:

$$\left[\begin{array}{ccc|c} 2 & 4 & -2 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right] \overset{M_1^{(1/2)}}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right] \overset{A_{1,2}^{(-3)}}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -1 & 3 & 1 \end{array}\right]$$

$$\overset{M_2^{(-1)}}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -3 & -1 \end{array}\right] \overset{A_{2,1}^{(-2)}}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & -3 & -1 \end{array}\right] = \text{rref}$$

EXAMPLE

Writing the general solution

$$\text{rref} = \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & -3 & -1 \end{array}\right]$$

The reduced row echelon form corresponds to

$$\begin{array}{ccccccc} x_1 & & & + & 5x_3 & = & 2 \\ & & x_2 & - & 3x_3 & = & -1 \end{array}$$

Solving for the leading variables x_1 and x_2 in terms of the free variable x_3:

$$\begin{array}{ccc} x_1 & = & 2 - 5x_3 \\ x_2 & = & -1 + 3x_3 \end{array}$$

Let $x_3 = t$. Then

$$x_1 = 2 - 5t, \qquad x_2 = -1 + 3t, \qquad x_3 = t, \qquad t \in \mathbb{R}$$

- So the system has infinitely many solutions, one for each real value of t.

EXAMPLE 5

Gauss–Jordan elimination on a 4 × 6 system

Solve

$$\begin{aligned} x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\ 5x_3 + 10x_4 + 15x_6 &= 5 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 &= 6 \end{aligned}$$

Solution — augmented matrix, then row reduce

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right]$$

Augmented Matrix

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

ref

EXAMPLE 5

Finishing the backward phase

Clearing above the leading 1's gives the reduced row echelon form

$$\left[\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

whose system, ignoring the zero row, is

$$\begin{aligned} x_1 + 3x_2 + 4x_4 + 2x_5 &= 0 \\ x_3 + 2x_4 &= 0 \\ x_6 &= \frac{1}{3} \end{aligned} \qquad (3)$$

Solve for the leading variables x_1, x_3, x_6 and assign $x_2 = r$, $x_4 = s$, $x_5 = t$:

$$x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = 1/3$$

EXAMPLE

Back-substitution on a triangular system

For hand computation Gauss–Jordan is convenient; for large systems on a computer it is usually cheaper to stop at a row echelon form and back-substitute. From Example 5 the row echelon form gives

$$\begin{aligned} x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\ x_3 + 2x_4 + 3x_6 &= 1 \\ x_6 &= \frac{1}{3} \end{aligned}$$

Back-substitute from the bottom up

- The last equation gives $x_6 = \frac{1}{3}$.
- Substituting into the second: $x_3 = 1 - 2x_4 - 3\left(\frac{1}{3}\right) = -2x_4$.
- Substituting into the first: $x_1 = -3x_2 - 4x_4 - 2x_5$.
- Assigning $x_2 = r$, $x_4 = s$, $x_5 = t$ recovers the same general solution as Example 5.

PROCEDURE

Solving a linear system — the whole recipe

- Write the augmented matrix of the system.
- Row reduce to a row echelon form (the forward phase). Decide whether the system is consistent — if a row $[0 \quad \cdots \quad 0 \quad b]$ with $b \neq 0$ appears, STOP: there is no solution.
- Continue to the reduced row echelon form (the backward phase).
- Write down the system of equations belonging to that matrix, discarding zero rows.
- Solve each equation for its leading variable in terms of the free variables, then name the free variables as parameters.

> ALWAYS CHECK. Substitute your answer back into the ORIGINAL equations — with parentheses around every substituted value. A parametric answer should satisfy them for every value of the parameters.

DEFINITION

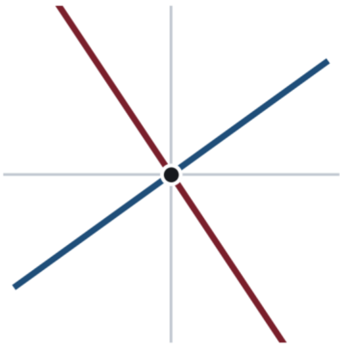
Homogeneous linear systems

A system is HOMOGENEOUS when every constant term is zero:

$$\begin{array}{ccccccc} a_{11}x_1 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & \cdots & + & a_{2n}x_n & = & 0 \\ \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & \cdots & + & a_{mn}x_n & = & 0 \end{array}$$

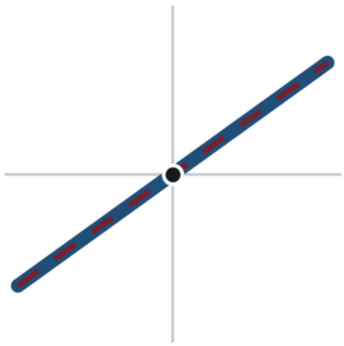
- Every homogeneous system is **consistent**, because $x_1 = x_2 = \cdots = x_n = 0$ is always a solution. This is the TRIVIAL SOLUTION; any other is a NONTRIVIAL SOLUTION.

Only the trivial solution



distinct lines through the origin

Infinitely many solutions



the lines coincide

> So a homogeneous system has exactly one of two outcomes: **only the trivial solution**, or **infinitely many solutions**.

EXAMPLE 6

A homogeneous system

The system of Example 5 with all right-hand sides replaced by 0 has augmented matrix differing only in its last column — and a column of zeros is unchanged by any row operation. So the reduced row echelon form is

$$\left[\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

giving $x_1 = -3x_2 - 4x_4 - 2x_5$, $x_3 = -2x_4$, $x_6 = 0$, and with $x_2 = r$, $x_4 = s$, $x_5 = t$,

$$x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = 0$$

- The trivial solution is the case $r = s = t = 0$.

THEOREM

Free variables and nontrivial solutions

THEOREM 1.2.1 (Free Variable Theorem for Homogeneous Systems)

If a homogeneous linear system has n unknowns, and if the reduced row echelon form of its augmented matrix has r nonzero rows, then the system has $n - r$ free variables.

THEOREM 1.2.2

A homogeneous linear system with more unknowns than equations has infinitely many solutions.

- Why: if $m < n$ then $r \leq m < n$, so $n - r \geq 1$ and there is at least one free variable.
- Theorem 1.2.2 is about HOMOGENEOUS systems only. A nonhomogeneous system with more unknowns than equations need not be consistent at all.

SUMMARY

Unit 1A in one slide

- A linear equation has the form $a_1x_1 + \cdots + a_nx_n = b$; a finite collection of them is a linear system.
- A system is consistent or inconsistent, and the number of solutions is always 0, 1, or infinitely many.
- The augmented matrix encodes the system; the three elementary row operations change the matrix but never the solution set.
- Row echelon form: leading 1's stepping right, zeros below. Reduced row echelon form: zeros below AND above.
- Gauss–Jordan = forward phase + backward phase. Gaussian elimination = forward phase, finished by back-substitution.
- Leading variables are solved for; free variables become parameters and give the general solution.
- Homogeneous systems are always consistent; with more unknowns than equations they have infinitely many solutions.

NEXT — Section 1.3: matrices and matrix operations.