1. Systems of Linear Equations and Matrices

MATH 244

In this Chapter we learn about:

For the topics covered in this Chapter students are expected to:

1. **Understand Linear Systems** and classify their possible solution sets.
2. **Perform Gaussian Elimination** to solve systems of linear equations.
3. **Master Matrix Operations** such as addition, multiplication, scalar multiplication, transpose, and trace.
4. **Understand Matrix Inverses**, their properties, and use them to solve systems of linear equations.
5. **Explain the role of Elementary Matrices** in row operations and in computing the inverse of a matrix.
6. **Identify properties of diagonal, triangular, and symmetric matrices**, and their advantages.
7. **Use determinants** to analyze matrix invertibility, system consistency, and to solve linear systems

1.1 Introduction to Systems of Linear Equations

A linear equation in *n* variables:

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$$

a_i: real-number coefficients

x_i: variables needed to be solved

b : real-number constant term

a_1: leading coefficient

x_1: leading variable

Notes:

(1) Linear equations have no variables involved in products, roots, trigonometric, exponential, or logarithmic functions.

(2) Variables appear only to the first power.

Example: Linear or Nonlinear

Linear (a) $3x + 2y = 7$

(b) $\frac{1}{2}x + y - \pi z = \sqrt{2}$ Linear

Linear (c) $x_1 - 2x_2 + 10x_3 + x_4 = 0$

(d) $(\sin \pi)x_1 - 4x_2 = e^2$ Linear

Nonlinear (e) $xy + z = 2$

product of variables

x is the exponent

(f) $e^x - 2y = 4$ Nonlinear

Nonlinear (g) $\sin x_1 + 2x_2 - 3x_3 = 0$

trigonometric function

(h) $\frac{1}{x} + \frac{1}{y} = 4$ Nonlinear

not the first power

For a linear equation in *n* variables: $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$

A solution is a sequence of values $x_1 = s_1, x_2 = s_2, \cdots, x_n = s_n$

So that $a_1s_1 + a_2s_2 + a_3s_3 + \cdots + a_ns_n = b$

Solution set is the set of all solutions of a linear equation.

※ In most cases, there are infinitely many solutions of a linear equation, so we need some methods to represent the solution set.

Example: Parametric representation of a solution set

$x_1 + 2x_2 = 4$ with a solution (2, 1), i.e., $x_1 = 2, x_2 = 1$

※ If you solve for x_1 in terms of x_2, you obtain

$x_1 = 4 - 2x_2$ (in this form, the variable x_2 is free)

By letting $x_2 = t$ (the variable t is called a parameter), you can represent the solution set as

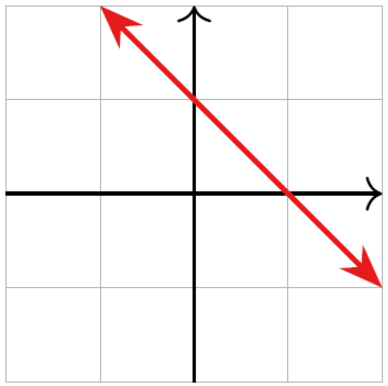
$x_1 = 4 - 2t,\ x_2 = t,\ t$ is any real number

The set representation for solutions: $\{(4-2t, t) \mid t \in R\}$

※ Choosing x_1 to be the free variable, the parametric representation of the solution set is $\{(s, 2 - \frac{s}{2}) | s \in R\}$

What does the solution set of a linear equation look like?

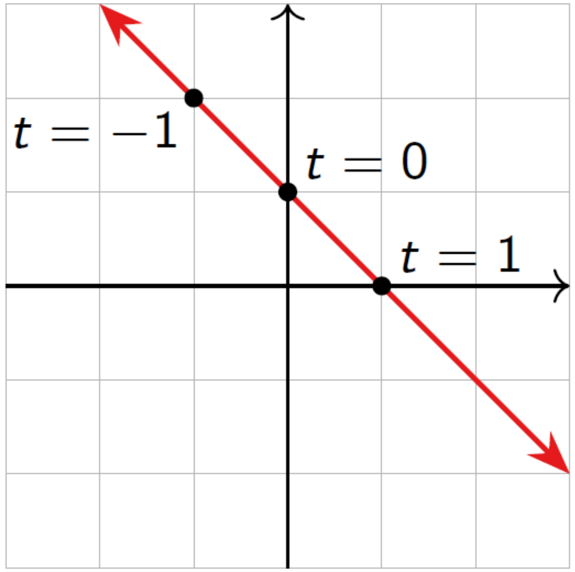
$x + y = 1 \rightsquigarrow$ a line in the plane: $y = 1 - x$
This is called the **implicit equation** of the line.



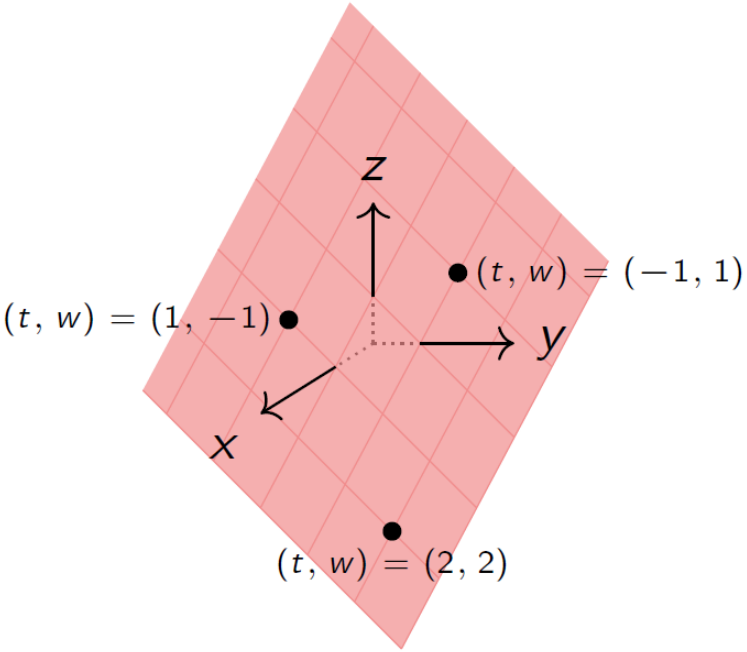
We can write the same line in **parametric form** in $\mathbf{R}^2$:

$$(x, y) = (t, 1 - t) \qquad t \text{ in } \mathbf{R}.$$

This means that every point on the line has the form $(t, 1 - t)$ for some real number t. Note we are using $\mathbf{R}$ to *label* the points on a line in $\mathbf{R}^2$.



What does the solution set of a linear equation look like?



$x + y + z = 1 \rightsquigarrow$ a plane in space:
This is the **implicit equation** of the plane.

Does this plane have a **parametric form**?

$$(x, y, z) = (t,\ w,\ 1 - t - w) \qquad t, w \text{ in } \mathbf{R}.$$

Note we are using $\mathbf{R}^2$ to *label* the points on a plane in $\mathbf{R}^3$.

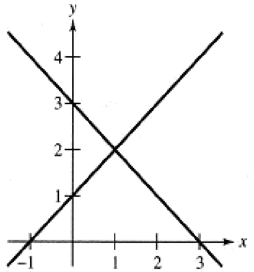
- A system of *m* linear equations in *n* variables:

$$\begin{array}{ccccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & \cdots & + & a_{3n}x_n & = & b_3 \\ & & & & \vdots & & & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

- A **solution** of a system of linear equations is a sequence of numbers $s_1, s_2, \ldots, s_n$ that satisfies equation in the system.
- **The solution set** of a system of linear equations is the collection of all solutions.
- **Solving** a system of linear equations means finding the solution set of the system.
- A system is call **consistent** if it has at least one solution and it is called **inconsistent** if it has no solution.

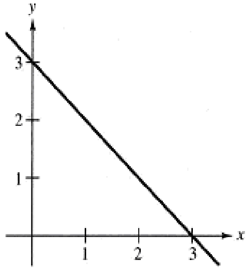
Example: Solution of a system of linear equations in 2 variables

(1)
$$\begin{aligned} x + y &= 3 \\ x - y &= -1 \end{aligned}$$
two intersecting lines



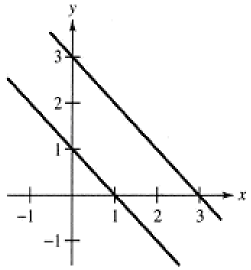
exactly one solution (consistent)

(2)
$$\begin{aligned} x + y &= 3 \\ 2x + 2y &= 6 \end{aligned}$$
two coincident lines



inifinitely many solution (consistent)

(3)
$$\begin{aligned} x + y &= 3 \\ x + y &= 1 \end{aligned}$$
two parallel lines



no solution (inconsistent)

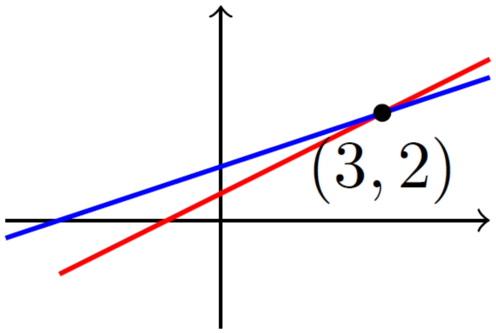
Example:

Consider the following systems. How are they different from each other?

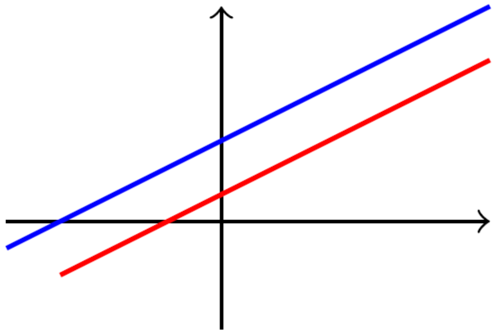
$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \end{aligned}$$

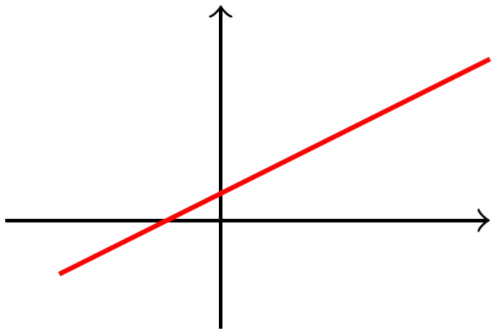
$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 1 \end{aligned}$$



non-parallel lines



parallel lines



identical lines

Example:

The system of 3 linear equations in 3 variables

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 4 & (1) \\ 8x_1 - x_2 + 2x_3 &= 0 & (2) \\ -x_1 - 2x_2 + 3x_3 &= 1 & (3) \end{aligned}$$

graphically represents three planes in $\mathbb{R}^3$.

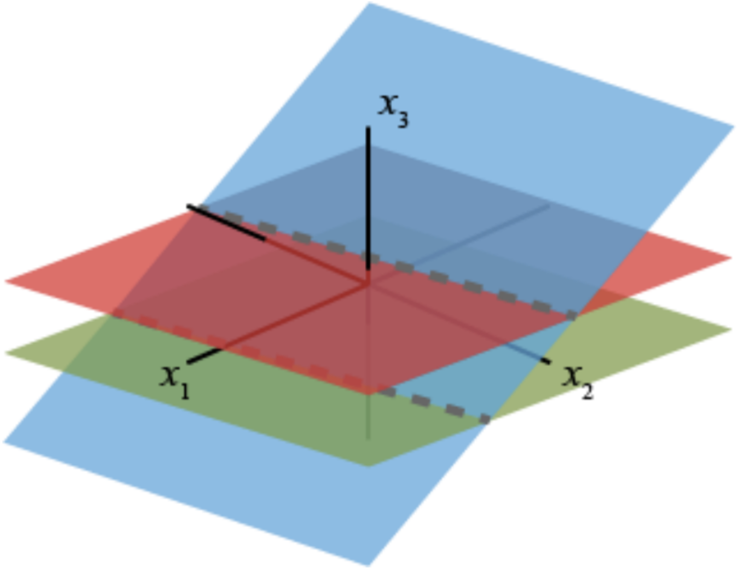
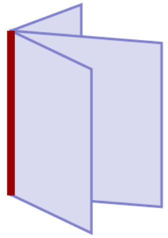
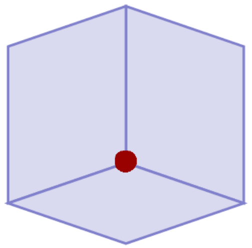
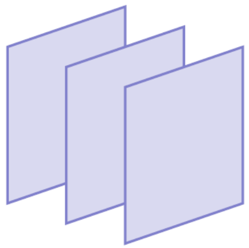


Image Description: There are two parallel and distinct planes, corresponding to equations (1) and (3). A third, non-parallel plane, corresponding to equation (2), intersects planes (1) and (3) separately.

From the diagram, we see that there is no point that lies on all three planes so the system is inconsistent.

Example:

An equation $a_1x_1 + a_2x_2 + a_3x_3 = b$ defines a plane in $\mathbb{R}^3$. The **solution** to a system of **three equations** is the set of intersections of the planes.

solution set	**sketch**	**number of solutions**
line		
point		
empty		

Example: Using **back substitution** to solve a system in **row-echelon form**

$$\begin{aligned} x - 2y &= 5 \qquad (1) \\ y &= -2 \qquad (2) \end{aligned}$$

※ The row-echelon form means that the system follows a stair-step pattern

※ The back substitution means to solve a system in reverse order.

Sol: By substituting $y = -2$ into Eq. (1), you obtain

$$x - 2(-2) = 5 \Rightarrow x = 1$$

The system has exactly one solution: $x = 1, y = -2$

Example: Using back substitution to solve a system in row-echelon form

$$\begin{aligned} x - 2y + 3z &= 9 \quad (1) \\ y + 3z &= 5 \quad (2) \\ z &= 2 \quad (3) \end{aligned}$$

Sol: Substitute $z = 2$ into (2), y can be solved as follows

$$\begin{aligned} y + 3(2) &= 5 \\ y &= -1 \end{aligned}$$

and substitute $y = -1$ and $z = 2$ into (1)

$$\begin{aligned} x - 2(-1) + 3(2) &= 9 \\ x &= 1 \end{aligned}$$

The system has exactly one solution:

$$x = 1, y = -1, z = 2$$

- **Equivalent Systems:** Two linear systems are called equivalent if they have the same solution set.

Notes: Each of the following **operations** on a system of linear equations produces an equivalent system

O1: Interchange two equations

O2: Multiply an equation by a nonzero constant

O3: Add a multiple of an equation to another equation

- **Gaussian elimination:**

A procedure to rewrite a system of linear equations to be in row-echelon form by using the above three operations

Example: Solve a system of linear equations (consistent system)

$$\begin{array}{rcrcrcrl} x & - & 2y & + & 3z & = & 9 & (1) \\ -x & + & 3y & & & = & -4 & (2) \\ 2x & - & 5y & + & 5z & = & 17 & (3) \end{array}$$

Sol: First, eliminate the *x*-terms in Eqs. (2) and (3) based on Eq. (1)

$(1)+(2) \rightarrow (2)$ (by O3)

$$\begin{array}{rcrcrcrl} x & - & 2y & + & 3z & = & 9 & \\ & & y & + & 3z & = & 5 & (4) \\ 2x & - & 5y & + & 5z & = & 17 & \end{array}$$

$(1)\times(-2)+(3) \rightarrow (3)$ (by O3)

$$\begin{array}{rcrcrcrl} x & - & 2y & + & 3z & = & 9 & \\ & & y & + & 3z & = & 5 & \\ & & -y & - & z & = & -1 & (5) \end{array}$$

Second, eliminate the (-*y*)-term in Eq. (5) based on Eq. (4)

$(4)+(5)\rightarrow(5)$ (by O3)

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2z &= 4 \qquad (6) \end{aligned}$$

$(6)\times\frac{1}{2}\rightarrow(6)$ (by O2)

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned}$$

Since the system of linear equations is expressed in its row-echelon form, the solution can be derived by the back substitution: $x=1,\ y=-1,\ z=2$ (only one solution)

Example: Solve a system of linear equations (inconsistent system)

$$\begin{array}{rcrcrcrr} x_1 & - & 3x_2 & + & x_3 & = & 1 & \quad (1) \\ 2x_1 & - & x_2 & - & 2x_3 & = & 2 & \quad (2) \\ x_1 & + & 2x_2 & - & 3x_3 & = & -1 & \quad (3) \end{array}$$

Sol:

$$(1) \times (-2) + (2) \rightarrow (2) \text{ (by O3)}$$

$$(1) \times (-1) + (3) \rightarrow (3) \text{ (by O3)}$$

$$\begin{array}{rcrcrcrr} x_1 & - & 3x_2 & + & x_3 & = & 1 & \\ & & 5x_2 & - & 4x_3 & = & 0 & \quad (4) \\ & & 5x_2 & - & 4x_3 & = & -2 & \quad (5) \end{array}$$

$$(4)\times(-1)+(5)\rightarrow(5)\ \text{(by O3)}$$

$$\begin{array}{ccccccc} x_1 & - & 3x_2 & + & x_3 & = & 1 \\ & & 5x_2 & - & 4x_3 & = & 0 \\ & & & & 0 & = & -2 \end{array}$$

(a false statement)

So, the system has no solution (an inconsistent system)

Example: Solve a system of linear equations (consistent system)

$$\begin{aligned} x_2 - x_3 &= 0 \quad (1)\\ x_1 \qquad - 3x_3 &= -1 \quad (2)\\ -x_1 + 3x_2 \qquad &= 1 \quad (3) \end{aligned}$$

Sol: $(1) \leftrightarrow (2)$ (by O1)

$$\begin{aligned} x_1 \qquad - 3x_3 &= -1 \quad (1)\\ x_2 - x_3 &= 0 \quad (2)\\ -x_1 + 3x_2 \qquad &= 1 \quad (3) \end{aligned}$$

$(1) + (3) \rightarrow (3)$ (by O3)

$$\begin{aligned} x_1 \qquad - 3x_3 &= -1\\ x_2 - x_3 &= 0\\ 3x_2 - 3x_3 &= 0 \quad (4) \end{aligned}$$

Since Equation (4) is the same as Equation (2), it is not necessary and can be omitted

$$\begin{array}{lllll} x_1 & & - 3x_3 & = & -1 \\ & x_2 & - x_3 & = & 0 \end{array}$$

$$\Rightarrow x_2 = x_3, \quad x_1 = -1 + 3x_3$$

Let $x_3 = t$, then $x_1 = 3t - 1$

$x_2 = t \qquad t \in R$

$x_3 = t$

This system has infinitely many solutions.

- A system of *m* equations in *n* variables:

$$\begin{array}{ccccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \ldots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \ldots & + & a_{2n}x_n & = & b_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & \ldots & + & a_{3n}x_n & = & b_3 \\ & & & & \vdots & & & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \ldots & + & a_{mn}x_n & = & b_m \end{array}$$

- **Augmented matrix**: formed by appending the constant-term vector to the right of the coefficient matrix of a system of linear equations

$$\left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ & & \vdots & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{array}\right] = [A \mid \mathbf{b}]$$

1.2 Gaussian Elimination

- Row-echelon form: (1), (2), and (3)
- Reduced row-echelon form: (1), (2), (3), and (4)

(1) Any rows consisting entirely of zeros must be at the bottom of the matrix.

(2) The first nonzero entry in any row must be 1, which is called as leading.

(3) The leading 1 in a higher row is to the left of the leading 1 in a lower row.

(4) Every column that contains a leading 1 has zeros everywhere else.

Example: Row-echelon form or reduced row-echelon form?

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$ (row-echelon form)

$$\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ (reduced row-echelon form)

$$\begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$ (row-echelon form)

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ (reduced row-echelon form)

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

Violate the second criterion

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$$

Violate the first criterion

- Gaussian elimination:

  The procedure for reducing a matrix to a **row-echelon form** by performing the three elementary row operations

- Gauss-Jordan elimination:

  The procedure for reducing a matrix to its **reduced row-echelon form** by performing the three elementary row operations

- Notes:

  (1) Every matrix has a unique reduced row-echelon form.

  (2) A row-echelon form of a given matrix is not unique (Different sequences of elementary row operations can produce different row-echelon forms).

Elementary row operations:

(1) Interchange two rows: $I_{i,j} \equiv R_i \leftrightarrow R_j$

(2) Multiply a row by a nonzero constant: $M_i^{(k)} \equiv (k)R_i \to R_i$

(3) Add a multiple of a row to another row: $A_{i,j}^{(k)} \equiv (k)R_i + R_j \to R_j$

Example: Gauss and Gauss-Jordan elimination algorithm

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix} \xrightarrow{I_{1,2}} \begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

The first nonzero column

Produce a leading 1

$$\xrightarrow{M_1^{(\frac{1}{2})}} \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix} \xrightarrow{A_{1,3}^{(-2)}} \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

Leading 1

Eliminate nonzero entries below the leading 1

Produce a leading 1

The first nonzero column

Submatrix

$$\xrightarrow{M_2^{(-\frac{1}{2})}} \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

Leading 1

Eliminate nonzero entries below the leading 1

$$\xrightarrow{A_{2,3}^{(-5)}} \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1/2 & 1 \end{bmatrix}$$

The first nonzero column

Produce a leading 1

Submatrix

$$\xrightarrow{M_3^{(2)}} \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Nonzero entries

Leading 1's

(row echelon form)

REF

$$\xrightarrow{A_{3,1}^{(-6)}\ A_{3,2}^{(\frac{7}{2})}} \begin{bmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{A_{2,1}^{(5)}} \begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

(reduced row echelon form)

RREF

Example: Solve a system by the Gauss-Jordan elimination method

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y \quad\quad &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

Sol: From the augmented matrix

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix} \xrightarrow{A_{1,2}^{(1)}\ A_{1,3}^{(-2)}} \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix} \xrightarrow{A_{2,3}^{(1)}} \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix} \xrightarrow{M_{3}^{(1/2)}} \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

(row-echelon form)

$$\xrightarrow{A_{3,2}^{(-3)}\ A_{3,1}^{(-3)}} \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{A_{2,1}^{(2)}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{aligned} x &= 1 \\ y &= -1 \\ z &= 2 \end{aligned}$$

(reduced row-echelon form)

Example: Solve a system by the Gauss-Jordan elimination method

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 0 \\ 3x_1 + 5x_2 &= 1 \end{aligned}$$

Sol: From the augmented matrix

$$\begin{bmatrix} 2 & 4 & -2 & 0 \\ 3 & 5 & 0 & 1 \end{bmatrix} \xrightarrow{M_1^{(\frac{1}{2})}} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 3 & 5 & 0 & 1 \end{bmatrix} \xrightarrow{A_{1,2}^{(-3)}} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -1 & 3 & 1 \end{bmatrix}$$

$$\xrightarrow{M_2^{(-1)}} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -3 & -1 \end{bmatrix} \xrightarrow{A_{2,1}^{(-2)}} \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

(row-echelon form) (reduced row-echelon form)

Then the system of linear equations becomes

$$\begin{aligned} x_1 \qquad\quad + 5x_3 &= 2 \\ x_2 - 3x_3 &= -1 \end{aligned}$$

It can be further reduced to

$$\begin{aligned} x_1 &= 2 - 5x_3 \\ x_2 &= -1 + 3x_3 \end{aligned}$$

Let $x_3 = t$, then

$$\begin{aligned} x_1 &= 2 - 5t, \\ x_2 &= -1 + 3t, \qquad t \in R \\ x_3 &= t, \end{aligned}$$

So, this system has infinitely many solutions

※ As an experiment to show that the reduced row-echelon form is unique, we perform a redundant elementary row operation of interchanging the first and second rows in advance

$$\begin{bmatrix} 2 & 4 & -2 & 0 \\ 3 & 5 & 0 & 1 \end{bmatrix} \xrightarrow{I_{1,2}} \begin{bmatrix} 3 & 5 & 0 & 1 \\ 2 & 4 & -2 & 0 \end{bmatrix} \xrightarrow{M_1^{(1/3)}} \begin{bmatrix} 1 & 5/3 & 0 & 1/3 \\ 2 & 4 & -2 & 0 \end{bmatrix}$$

$$\xrightarrow{A_{1,2}^{(-2)}} \begin{bmatrix} 1 & 5/3 & 0 & 1/3 \\ 0 & 2/3 & -2 & -2/3 \end{bmatrix} \xrightarrow{M_2^{(3/2)}} \begin{bmatrix} 1 & 5/3 & 0 & 1/3 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

(row-echelon form)

$$\xrightarrow{A_{2,1}^{(-5/3)}} \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

(reduced row-echelon form)

※ Comparing with the previous results, we can infer that it is possible to derive different row-echelon forms, but there is a unique reduced row-echelon form for each matrix

Notes:

1. A linear system will not have a solution if and only if it has an impossible equation of the form $0 = b$ and $b \neq 0$.
2. An equation of the form $0 = 0$ is satisfied by any values of the variables and hence the solutions will depend on the other equations.
3. If the system has no impossible equation, then
   - it has a unique solution if number of variables = the number of nonzero equations in REF.
   - it has infinitely many solutions if number of variables is more than the nonzero equations in REF.
   - The number of variables cannot be fewer than the nonzero equations in REF.